Mathematical Modelling I
Calculus – Definite integrals & applications

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Week Five

Birmingham City University, Engineering and Built Environment

What will you learn

These lessons will be mostly, obviously, maths :

- what it is - definite integrals
- why it works - physics and eng.
- how it works - practice !

I will try to explain in detail (maybe too much). *Please* let me know if you do not understand something.

The weight-lifter

He is pulling the weight over his head.
The associated potential energy is :

$$E_w = mgh$$

With $g = 10m/s^2, m = 200kg$ and $h = 2m$, we have :

$$E_w = 4000J$$

The girl

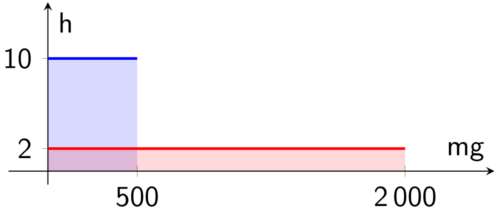
She is moving to the third floor The associated potential energy is :

$$E_w = mgh$$

With $g = 10m/s^2, m = 50kg$ and $h = 10m$, we have :

$$E_w = 5000J$$

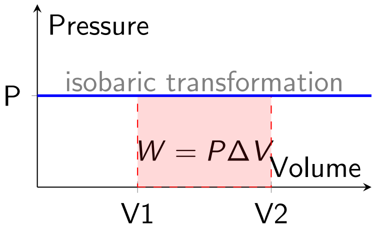
How to represent the energy ?



The energy being $E = mgh$, it is actually the area under the curve.
It is convenient to compare similar efforts.

Areas in engineering

Calculating an area is fundamental in engineering :



The area of the path in the Pressure-Volume space is the Work done

In the following, we will see how to calculate the area under a curve, and the underlying mathematics.

Areas in engineering

Calculating an area is fundamental in engineering :

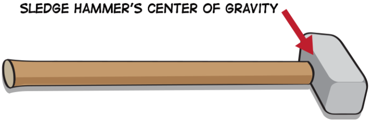
L'Oceanografic (Valencia, Spain), Gabaldon/Wikimedia Commons

How much paint to paint a roof

In the following, we will see how to calculate the area under a curve, and the underlying mathematics.

Areas in engineering

Calculating an area is fundamental in engineering :



NASA

calculating the center of mass (in 2D)

In the following, we will see how to calculate the area under a curve, and the underlying mathematics.

Areas in engineering

Calculating an area is fundamental in engineering :

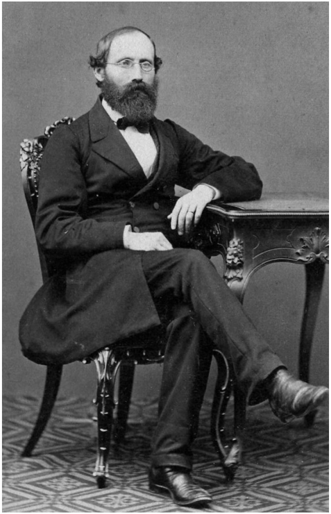
Many more subject are related to similar notions, for instance calculating volumes, the average of a complex function, etc.

In the following, we will see how to calculate the area under a curve, and the underlying mathematics.

Riemann

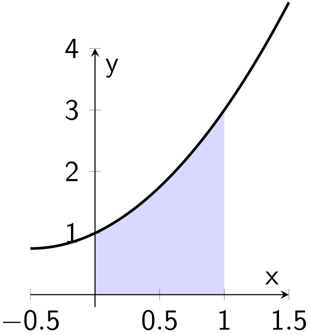
" The greatest strategy is doomed if it's implemented badly. "

Riemann



Area under the curve

Let's approximate the area under a curve ($f(x) = x^2 + x + 1$) and the axis, between 0 and 1 :



A way to do so is to add some small rectangles, with areas

$$\mathcal{A}_1 = f(0.0) \times \frac{1}{5}.$$

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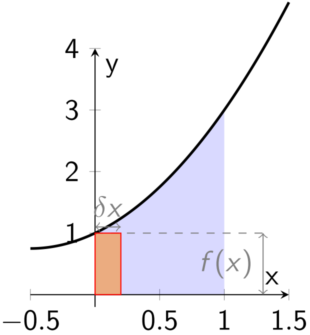

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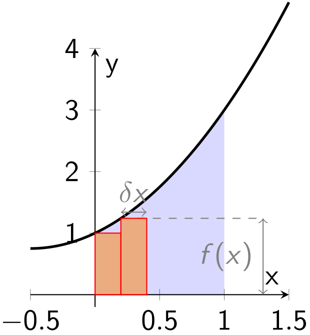
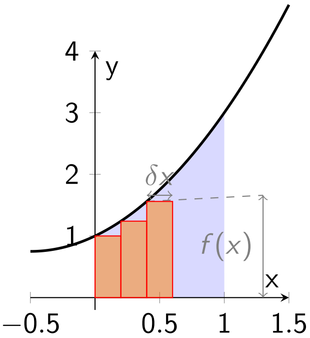


A way to do so is to add some small rectangles, with areas

$$\mathcal{A}_2 = f(0.2) \times \frac{1}{5}.$$

Area under the curve

Let's approximate the area under a curve ($f(x) = x^2 + x + 1$) and the axis, between 0 and 1 :

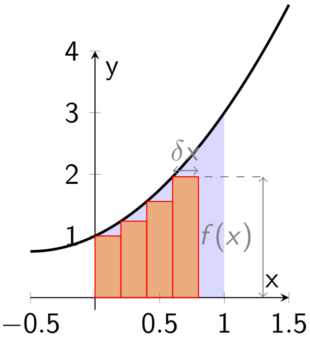


A way to do so is to add some small rectangles, with areas

$$\mathcal{A}_3 = f(0.4) \times \frac{1}{5}.$$

Area under the curve

Let's approximate the area under a curve ($f(x) = x^2 + x + 1$) and the axis, between 0 and 1 :

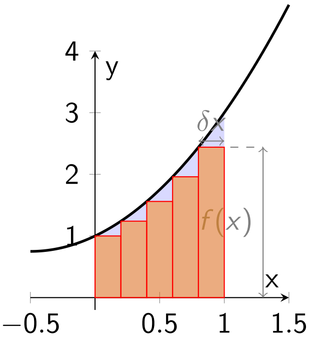


A way to do so is to add some small rectangles, with areas

$$\mathcal{A}_4 = f(0.6) \times \frac{1}{5}.$$

Area under the curve

Let's approximate the area under a curve ($f(x) = x^2 + x + 1$) and the axis, between 0 and 1 :



A way to do so is to add some small rectangles, with areas

$$\mathcal{A}_5 = f(0.8) \times \frac{1}{5}.$$

Details on the rectangles

What is the area $\mathcal{A}_i$ of the ith rectangle ?

$$\text{area of a rectangle} = \text{width} \times \text{height}$$

- width of $\mathcal{A}_i$:
  Each of them have a width of $\delta x = \frac{1-0}{n}$, where
  1. n is the number of rectangles (5 in the previous figure)
  2. 1 and 0 from "1-0" are the right and left limits of the area

  If the rectangle starts in x_i and ends in x_{i+1}, then we have $\delta x = x_{i+1} - x_i$
  We can rewrite it as $x_i = (i-1)\delta x$
- height of $\mathcal{A}_i$:
  the height is $f(x_i)$,

and its area is hence

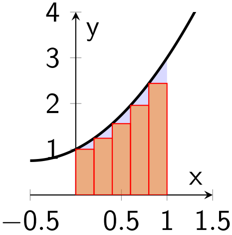
$$\mathcal{A}_i = f(x_i) \times \delta x$$

Approximation of the Area $\mathcal{A}$

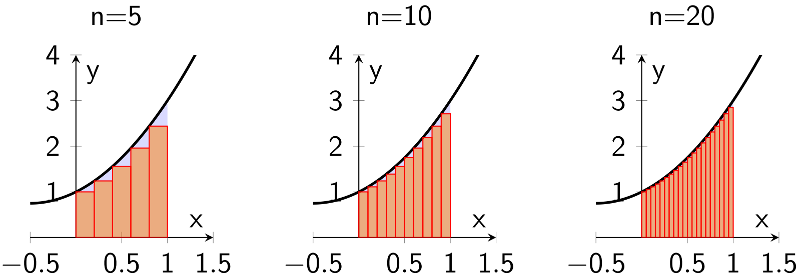
We have approximated the area by using and summing tiny rectangles.

The full area $\mathcal{A}$ is roughly the sum of the A_i :

$$\mathcal{A} \approx \sum_{i=1}^{n} \mathcal{A}_i$$



How does n affect the approximation ?

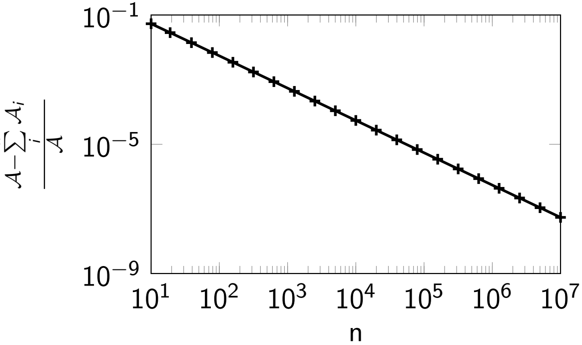


$$\frac{\mathcal{A}-\sum_i^N \mathcal{A}_i}{\mathcal{A}} = 11\%$$

$$\frac{\mathcal{A}-\sum_i^N \mathcal{A}_i}{\mathcal{A}} = 5\%$$

$$\frac{\mathcal{A}-\sum_i^N \mathcal{A}_i}{\mathcal{A}} = 2\%$$

from $n = 10$ to $n = 10^7$



A conclusion

It seems that, when n goes to ∞, the sum of the rectangles converges to the area under the curve $\mathcal{A}$!

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Or, rewritten in a mathematical language :

$$\mathcal{A} = \lim_{n\to\infty} \sum_{i=1}^{n} f\big((i-1)\delta x\big)\delta x$$

Remember :

- $f\big((i-1)\delta x\big)$ *is the height in the ith point*
- $\delta x = \dfrac{1}{n}$ *is the width*
- *and hence the product is the area* $\mathcal{A}_i$

A conclusion

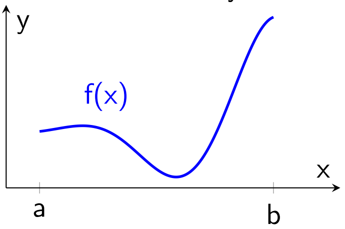
It seems that, when n goes to ∞, the sum of the rectangles converges to the area under the curve $\mathcal{A}$!
Or, rewritten in a mathematical language :

$$\mathcal{A} = \lim_{n \to \infty} \sum_{i=1}^{n} f\left((i-1)\delta x\right)\delta x$$

In the next slides, we will just generalize and give more strength to this idea !

Considerations

In the following, we are interested in functions that are defined on $[a, b] \in \mathbb{R}$.
a and b can be any number, and we suppose that $a < b$.

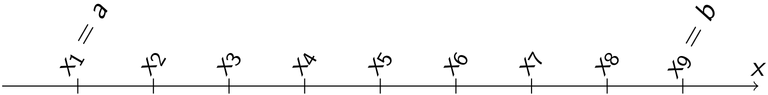


Set of points

Let's have a set of points $\{x_i\}_{1\leq i\leq n}$, such as :

- $x_1 = a$
- $x_{i+1} > x_i$
- $x_n = b$

We are just dividing $[a, b]$!
The set of point in called a *partition* of $[a, b]$ of *size n*.



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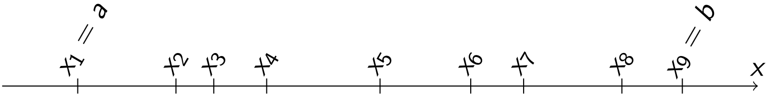
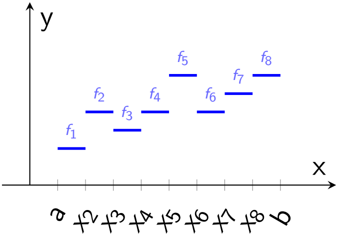


Piecewise constant function

Let's consider functions that are constant on each of the intervals $[x_i, x_{i+1}]$ of a partition. It means, that if $x \in [x_i, x_{i+1}]$,

$$f(x) = f_i$$



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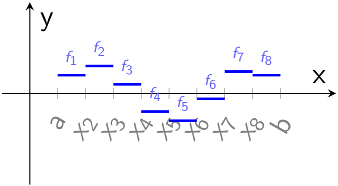
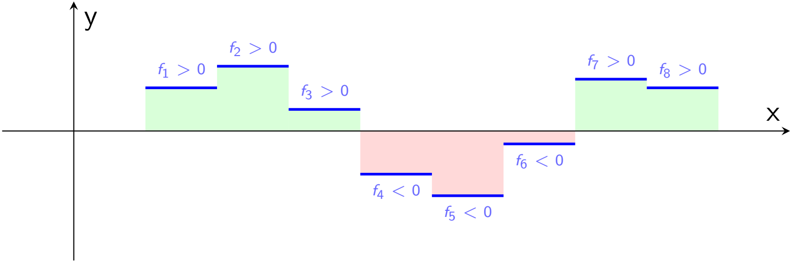


Area under the curve

We can now define what is the area under a piecewise function.
Let's say it is *negative* if $f_i < 0$.



The area under piecewise function is then :

$$\mathcal{A} = \sum_{i=1}^{n} f_i \times (x_{i+1} - x_i)$$

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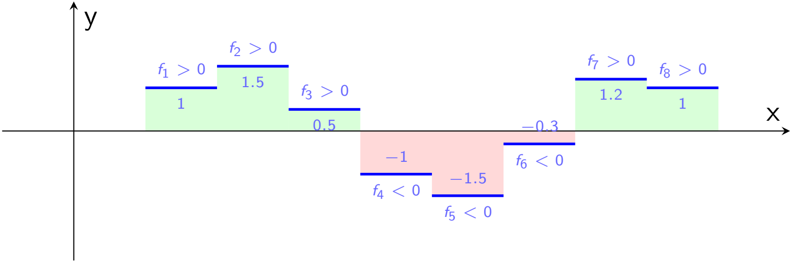

The area under piecewise function is then :

$$\text{if } x_{i+1} - x_1 = 1$$

$$\mathcal{A} = 1 + 1.5 + 0.5 - 1 - 1.5 - 0.3 + 1.2 + 1 = 2.4$$

Integral for piecewise functions

We actually define the integral for piecewise function as the area under the curve :

$$\int_a^b f(x)dx = \sum_{i=1}^{n} f_i \times (x_{i+1} - x_i)$$

This is the *Cauchy-Riemann integral* of a piecewise function. It is also the *definite integral* of a function between a and b.

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Why definite ?

- We have seen *indefinite integrals* :

$$\int f(x), \quad \textit{for instance} \int \sin(x)dx = -\cos(x) + C$$

It is a function. Also, there is no number around the $\int$ sign.

- We have now *definite integrals* :

$$\int_a^b f(x)dx$$

They are definite between two points, a and b. It is associated with a value. As long as a and b are known, it is *not* a function.

Why definite ?

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$$\int f(x), \quad \textit{for instance} \int \sin(x)dx = -\cos(x) + C$$

It is a function. Also, there is no number around the $\int$ sign.

Let's see why we are using the same sign $\int$!

$$\int_a^b f(x)dx$$

They are definite between two points, a and b. It is associated with a value. As long as a and b are known, it is *not* a function.

Approximation of a continuous function

The previous integral definition is only for a piecewise constant function.
It can easily be extended to continuous functions : by following the spirit of first example !

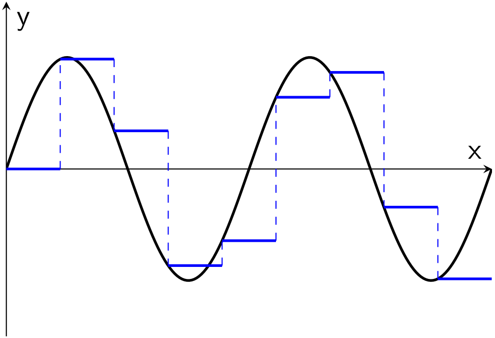
Approximation of a continuous function

First, we approximate a continuous function with a piecewise constant function.

- chose n and construct the partition $\sigma_n = \{x_1, \ldots, x_n\}$ with :
  - $x_1 = a, x_n = b$
  - $\delta x = \dfrac{b-a}{n}$
  - $x_i = x_1 + (i-1)\delta x$
- approximate a continuous function f with a piecewise constant function $\bar{f}_n$, on a partition σ_n, with $\bar{f}_n(x) = f(x_i)$ if $x \in [x_i, x_{i+1}]$.

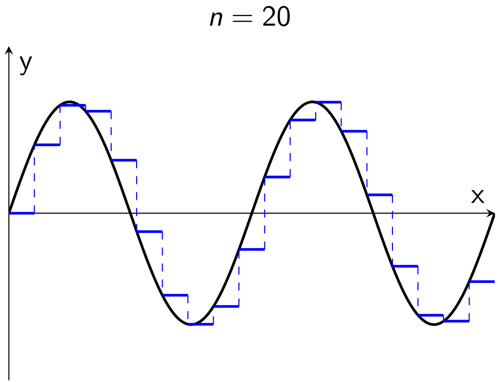
Approximation of a continuous function : illustration

$n = 10$



Note how $f_i = f(x_i)$.

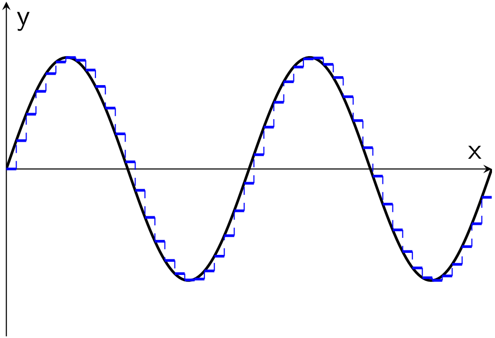
Approximation of a continuous function : illustration



Note how $f_i = f(x_i)$.

Approximation of a continuous function : illustration

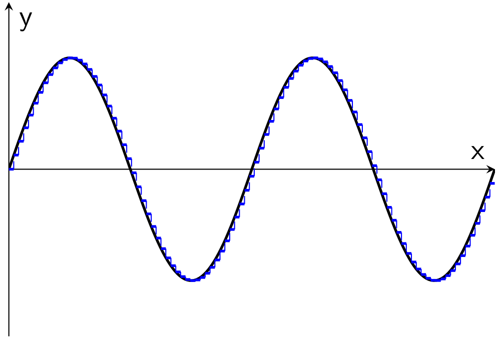
$n = 50$



Note how $f_i = f(x_i)$.

Approximation of a continuous function : illustration

$n = 100$



Note how $f_i = f(x_i)$.

Going to the limit

Now that we have the approximation $\bar{f}_n$ of f :

- estimate the integral of the piecewise function $\bar{I}_n = \int\limits_a^b \bar{f}_n(x)dx = \sum\limits_{i=1}^{n} f_i \delta x.$
- The integral I of f is the limit of I_n when $n \to \infty$:

$$I = \lim_{n\to\infty} \sum_{i=1}^{n} f_i \delta x$$

I is noted $\int\limits_a^b f(x)dx$ and is called the integral of f on $[a, b]$.

Going to the limit

Now that we have the approximation $\bar{f}_n$ of f :

- estimate the integral of the piecewise function
-

the integral of f on $[a, b]$ is :

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\big(a + i(a-b)/n\big)\delta x$$

It is the *sum* of small elements of width dx.

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A real problem in architecture

Say we want to find the surface of glass needed to cover the front of the Oceanografic, a gigantic oceanonarium :

L'Oceanografic (Valencia, Spain), Gabaldon/Wikimedia Commons

The front is a parabola, the surface is hence related to the integral of a quadratic equation, i.e., $f(x) = x^2$.

Integrate $\int_a^b x^2 dx$

Let's integrate the function $f(x) = x^2$ between a and b.
It should not be that long, right ?

Integrate $\int_a^b x^2 dx$

Let's integrate the function $f(x) = x^2$ between a and b.
It should not be that long, right ?

First, let's use the formula defining the integral :
$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i(a-b)/n\right)\delta x$. With $f(x) = x^2$, we have :

$$\int_a^b x^2 dx = \lim_{n\to\infty} \left\{ \sum_{i=1}^{n} \left[a + i\left(\frac{b-a}{n}\right)\right]^2 \times \left(\frac{b-a}{n}\right) \right\}$$

Integrate $\int_a^b x^2 dx$

Maybe, it will not be that easy.

photo by Mathew Schwartz

Integrate $\int_a^b x^2 dx$

Maybe, it will not be that easy.
Let's continue ?

$$
\begin{aligned}
\int_a^b x^2 dx &= \lim_{n\to\infty}\left\{\sum_{i=1}^{n}\left[a^2+2ai\frac{b-a}{n}+i^2\left(\frac{b-a}{n}\right)^2\right]\times\frac{b-a}{n}\right\}\\
&= \lim_{n\to\infty}\left\{\sum_{i=1}^{n}\left[a^2\frac{b-a}{n}+2ai\left(\frac{b-a}{n}\right)^2+i^2\left(\frac{b-a}{n}\right)^3\right]\right\}\\
&= \lim_{n\to\infty}\left\{a^2(b-a)+2a\left(\frac{b-a}{n}\right)^2\left[\sum_{i=1}^{n} i\right]+\left(\frac{b-a}{n}\right)^3\left[\sum_{i=1}^{n} i^2\right]\right\}\\
&= \lim_{n\to\infty}\left\{a^2(b-a)+2a\left(\frac{b-a}{n}\right)^2 I_1+\left(\frac{b-a}{n}\right)^3 I_2\right\}
\end{aligned}
$$

with

$$I_1=\sum_{i=1}^{n} i,\; I_2=\sum_{i=1}^{n} i^2$$

Integrate $\int_a^b x^2 dx$

Maybe, it will not be that easy.
Urgh.

photo by Marc Lopez

Integrate $\int_a^b x^2 dx$

Or, we have

$$\begin{aligned} I_1 &= \sum_{i=1}^{n} i \\ &= \frac{n(n+1)}{2} \end{aligned}$$

and

$$\begin{aligned} I_2 &= \sum_{i=1}^{n} i^2 \\ &= \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

Integrate $\int_a^b x^2 dx$

We can now replace I_1 and I_2

$$\begin{aligned}
\int_a^b x^2 dx &= \lim_{n\to\infty} \left\{ a^2(b-a) + 2a\left(\frac{b-a}{n}\right)^2 I_1 + \left(\frac{b-a}{n}\right)^3 I_2 \right\} \\
&= \lim_{n\to\infty} \left\{ a^2(b-a) + \cancel{2}a\left(\frac{b-a}{n}\right)^2 \frac{n(n+1)}{\cancel{2}} \right. \\
&\quad \left. + \left(\frac{b-a}{n}\right)^3 \frac{n(n+1)(2n+1)}{6} \right\} \\
&= \lim_{n\to\infty} \left\{ a^2(b-a) + a(b-a)^2 \frac{n(n+1)}{n^2} \right. \\
&\quad \left. + \frac{(b-a)^3}{6} \frac{n(n+1)(2n+1)}{n^3} \right\}
\end{aligned}$$

Or, the limit of a sum is the sum of the limits.

Integrate $\int_a^b x^2 dx$

Breaking down the expression :

$$\lim_{n\to\infty} a^2 (b-a) = a^2 (b-a)$$

$$\lim_{n\to\infty} a (b-a)^2 \frac{n(n+1)}{n^2} = a (b-a)^2$$

$$\lim_{n\to\infty} \frac{(b-a)^3}{6} \frac{n(n+1)(2n+1)}{n^3} = \frac{(b-a)^3}{3}$$

Integrate $\int\limits_a^b x^2 dx$

So, we have :

$$
\begin{aligned}
\int\limits_a^b x^2 dx &= a^2 (b-a) + a (b-a)^2 + \frac{(b-a)^3}{3} \\
&= a^2 b - a^3 + a \left(b^2 - 2ab + a^2\right) + \frac{b^3 - 3ab^2 + 3a^b - a^3}{3} \\
&= a^2 b - \cancel{a^3} + ab^2 - 2a^2 b + \cancel{a^3} + \frac{b^3 - 3ab^2 + 3a^b - a^3}{3} \\
&= \frac{\cancel{3ab^2} - \cancel{3a^2 b} + b^3 - \cancel{3ab^2} + \cancel{3a^b} - a^3}{3}
\end{aligned}
$$

and, finally !

$$\int\limits_a^b x^2 dx = \frac{1}{3} b^3 - \frac{1}{3} a^3$$

Why even bothering ?

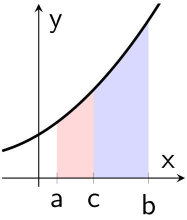
That was... not fun.

For that reason, there is a few theorems that will link *indefinite integral, definite integral and derivation.*

Additive property

Let's have $c \in [a, b]$.
Because the integral is the area under the curve, and the sum of the areas of two adjacent regions $[a, c]$ and $[c, b]$ is equal to the area of both regions combined $[a, b]$, we have

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$$



Reversal limits property

Also, integration in a backward direction gives the opposite of the forward integration :

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

The proof comes from the Riemann sum :

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f_i \delta x$$

If a and b are switched, then dx is $x_i - x_{i+1}$. Changing the sign gives the result. Et voilà !

Linearity property

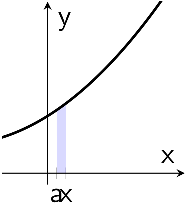
From the Riemann sum, we have :

$$\int_b^a \alpha f_1(x) + \beta f_2(x) dx = \alpha \int_a^b f_1(x) dx + \beta \int_a^b f_2(x) dx$$

First theorem : define F

Let's have x in $[a, b]$.
Then, we define a function F as the integral of f between a and x :

$$F(x) = \int_a^x f(t)dt$$



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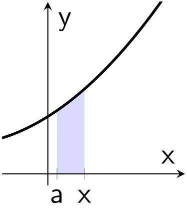

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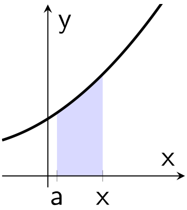

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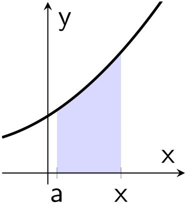

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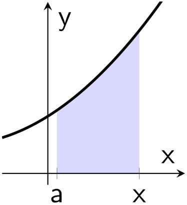

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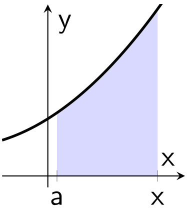

$F(x)$ is the blue area.

First fundamental theorem of calculus

We now can express the first theorem of calculus :

$$\text{If } F(x) = \int_a^x f(t)dt$$
$$\text{then : } F'(x) = f(x)$$

F is an *antiderivative* of f.

First fundamental theorem of calculus

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then : $F'(x) = f(x)$

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First theorem : sketch of proof

Let's have h :

$$\begin{aligned} F(x+h) - F(x) &= \int_a^{x+h} f(t)dt - \int_a^x f(t)dt \\ &= \int_x^{x+h} f(t)dt \end{aligned}$$

using the previous approximations with piecewise constant, $\int_x^{x+h} f(t)dt$ *is roughly* $f(x) \times h$*. It means that*

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x)$$

Et voilà !

Second fundamental theorem of calculus

If F is an antiderivative of f then :

$$\int_a^b f(t)dt = F(b) - F(a)$$

Second fundamental theorem of calculus

If F is an antiderivative of f then :

$$\int_a^b f(t)dt = F(b) - F(a)$$

Second theorem : proof

Let's have $G(x) = \int_a^x f(x)dx$ *an antiderivative of* f*. We know that* $(F - G)' = 0$*, hence* $G = F + C$ *where* C *is a constant. With* $x = a$*, we have :*

$$\int_a^a f(x)dx = G(a) = F(a) + C$$

Or $\int_a^a (f(x)dx = 0$*, which means* $C = -F(a)$*. finally,* $\int_a^b f(x)dx = G(b) = F(b) - F(a)$*.*
Et voilà !

Why these theorems are so important ?

- The definition allows us to break down calculations in small parts

  the area is a sum of small areas

- It links definite integrals and indefinite integrals !

  $F(x) = \int\limits_a^x f(x)dx$ *is an antiderivative of* f

- If *any antiderivative* F so $F' = f$ is known, then it is possible to calculate $\int_a^x f(t)dt$ for any x.

  it is $F(x) - F(a)$

- It has *so many* applications. Let's see !

  Mainly because of the first point !

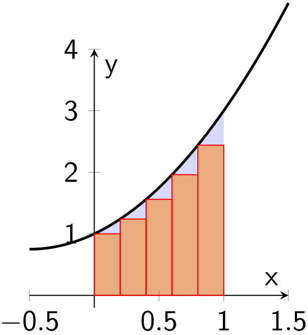
Cavalieri

" An area is considered as constituted by an indefinite numbers of parallel segments. "

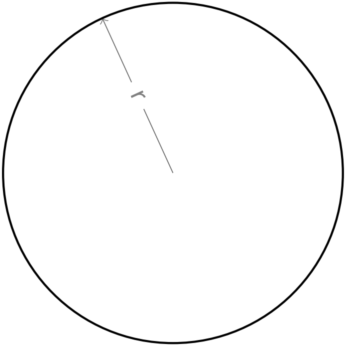
Early 17th-century Italian mathematician Bonaventura Cavalieri.

What we have seen

It is *very* valid to consider a small element dV of a problem and then to add all these elements, as $V = \int dV$ to identify the final value.
For instance, for an area, we have considered small areas and we have summed them to get the entire area :

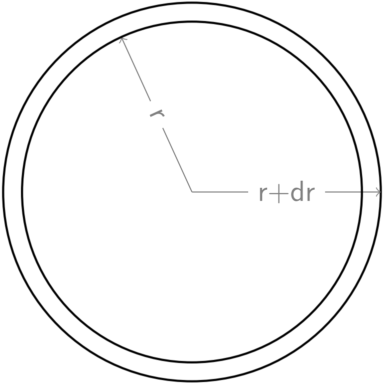


From the perimeter to the disk



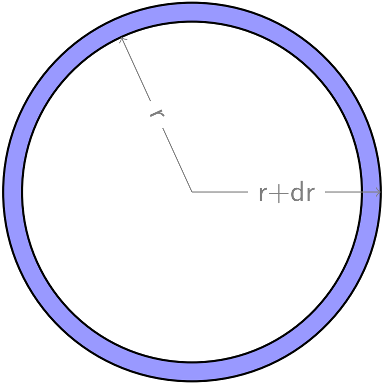
The perimeter of the circle is $2\pi r$.

From the perimeter to the disk



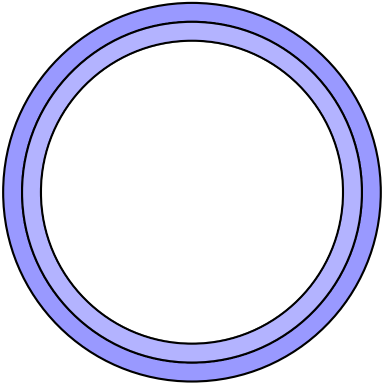
We consider a second circle of radius $r + dr$.

From the perimeter to the disk



If dr is small, the area δA is $2\pi r dr$.

From the perimeter to the disk



We can sum up all the δA to get the area.

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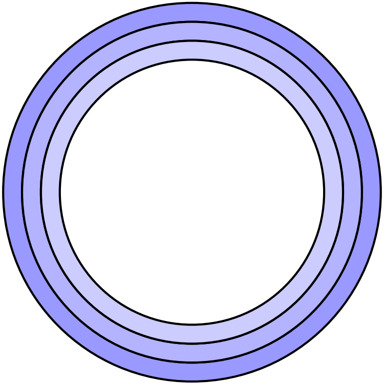


We can sum up all the δA to get the area.

From the perimeter to the disk

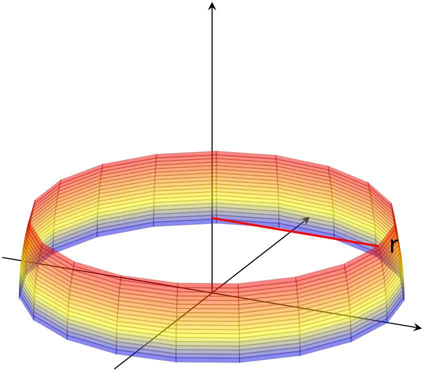
What is the perimeter of a circle of diameter r ? $P = 2\pi r$ Then, how to calculate the surface of a disk ? It is the sum of small bands of width dr :

$$S = \int_0^R 2\pi r dr$$

And the application gives

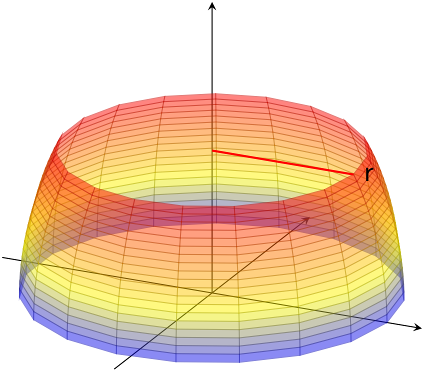
$$S = [\pi r^2]_0^R = \pi R^2$$

From the disk to the volume



A sphere, of radius R, can be approximated *stacked* disks. color, red is of radius r)

From the disk to the volume



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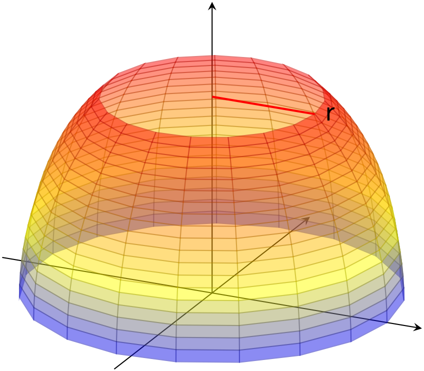

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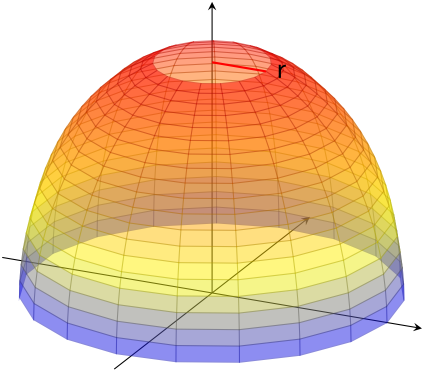

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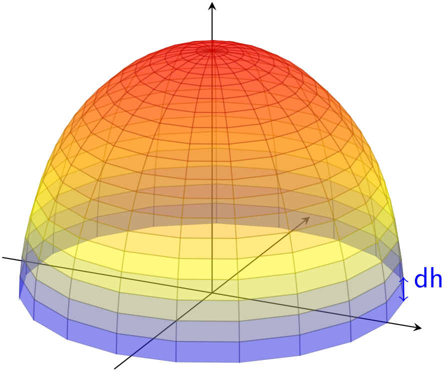

A sphere, of radius R, can be approximated *stacked* disks. color, red is of radius r) of height dh.

From the disk to the volume

How to calculate the volume of a sphere ?

Half the sphere is a sum of small disks of width dh :

$$dV = \pi r^2 dh, \frac{1}{2} V = \int_0^R \pi r^2 dh$$

how is r related to h ? We have $R^2 = r^2 + h^2$, and hence $r^2 = R^2 - h^2$.

From the disk to the volume

Combining both $r^= R^2 - h^2$ and $\frac{1}{2}V = \int_0^R \pi r^2 dh$ leads to :

$$\begin{aligned}\frac{1}{2}V &= \int_0^R \pi(R^2 - h^2)dh \\ &= \pi \int_0^R R^2 dh - \pi \int_0^R h^2 dh \\ &= \pi[R^2 h]_0^R - \pi[\frac{1}{3}h^3]_0^3 \\ &= \pi R^3 - \pi\frac{1}{3}R^3 \\ &= \pi\frac{2}{3}R^3\end{aligned}$$

and hence the celebrated $V = \frac{4}{3}\pi R^3$!

What if it was a cylinder instead ?

The volume is still stacked pancakes. But now, the radius does not depend on the height.

What if it was a cylinder instead ?

The volume is still stacked pancakes. But now, the radius does not depend on the height.
It means that

$$\begin{aligned} V &= \int_0^H \pi R^2 dh \\ &= \pi R^2 \int_0^H dh \\ &= \pi R^2 [h]_0^H \\ &= \pi R^2 H \end{aligned}$$

and hence the celebrated $V = \pi R^2 H$!

Work done by on a piston

What is the work done by a piston ? Let's think first of a piston. The element of work dW when the piston moves from dx is

$$dW = Fdx$$

What is F ? It comes from the pressure the gas exerces on the piston. If the piston has an area A, and if the pressure is p, then

$$dW = pAdx$$

Work done in general

Adx is the variation in volume in the piston chamber ! It is, practically, da. And hence, we can generalize to :

$$dW = pdv$$

. It means that the work from a gas, when the volume changes from V_1 to V_2 is :

$$W = \int_{V_1}^{V_2} pdv$$

Work done : case of a perfect gas

We now need to replace p with a function of v.
Let's suppose that we have the law of perfect gas, for an isotherm compression : $pv = c$ where c is a constant. It means that $p = \frac{c}{v}$, and the equation from work becomes :

$$W = \int_{V_1}^{V_2} \frac{c}{v} dv$$

The work is then :

$$W = [c \ln v]_{v_1}^{V_2}$$

and, finally :

$$W = c \ln \frac{V_2}{V_1}$$

Cylinder stress

$$\text{Rp} = \text{cste drP} + \text{dpR} = 0$$

$$\text{Stress P} \int_{P_0}^{P_1} \frac{dP}{a - P} = 2 \int_{r_0}^{r_1} \frac{dr}{r}$$

Sir Michel Atiyah

" I have a proof of the Riemann hypothesis ! "

2018 Heidelberg Laureate Forum (HLF)

Does Riemann's name sounds familiar ?

A few days ago (late September 2018), Atiyah gave a lecture in Germany in which he stated he proved the Riemann hypothesis.

Roughly speaking, the Riemann hypothesis explains how prime number are distributed (nothing to do with integrals...).
It is utterly important for cryptography and hence for the safety of our phones, emails and bank accounts !
Also, the one who will solve this hypothesis will win 1 million $!

Though Atiyah is supposed to be a "math wizard" (quote from John Allen Paulos, a professor of mathematics at Temple University in Philadelphia, USA), there is serious doubts about the proof.