Mathematics for Engineers–ENG 3009, 2018-2019

Introduction to law of Sines and Cosines

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As always, please free to refer to the book [Croft and Davidson, 2016] for details.

I Introduction

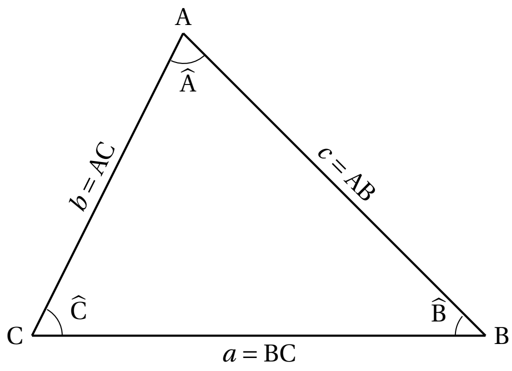


Figure 1: Here is a random triangle.

We will note

- the vertices with capital letters
- the sides with lower case letters
- Side a is opposite the angle $\widehat{A}$ and so on.
- the distance between two points A and B: AB.
- the angle between the line (BC) and the line (AC) either:
  - $\widehat{BCA}$
  - $\widehat{ACB}$
  - $\widehat{C}$

    note that C, as the summit, is always in the middle

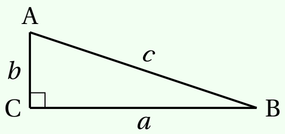
II Pythagoras theorem

Figure 2: You can take the first two pieces of chocolate, or the third one. Which one will you choose ?

Def.

PYTHAGORAS THEOREM: The Greek mathematician Pythagoras discovered, (or is credited with having discovered), that:

- ➔ in a right angled triangle
- ➔ the square of the hypotenuse is equal to the sum of the squares of the other two sides.



Note that the hypotenuse is the longest side (c).
It means, in the triangle above:

$$a^2 + b^2 = c^2$$

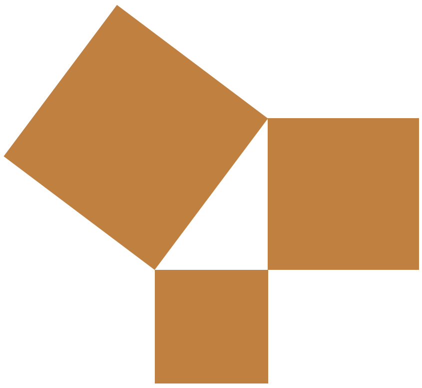
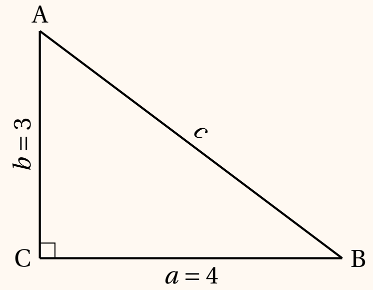


Figure 3: Following Pytaghoras theorem, it means that It does not matter !

Main Example



What is c ?
ABC is rectangle. Hence, we have that the hypothenuse is c. We can apply Pythagoras theorem:

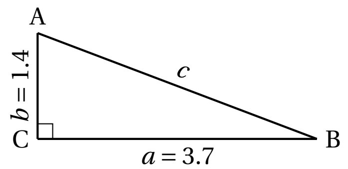
$$a^2 + b^2 = c^2$$

or, we have that $a = 4, b = 3$. By substitution:

$$\begin{aligned} c^2 &= 4^2 + 3^2 \\ &= 16 + 9 \\ &= 25 \end{aligned}$$

We now know c^2. To get c, we just have to take its square root, and hence $c = \sqrt{25} = 5$.

Here is a new triangle. Again, what is c?



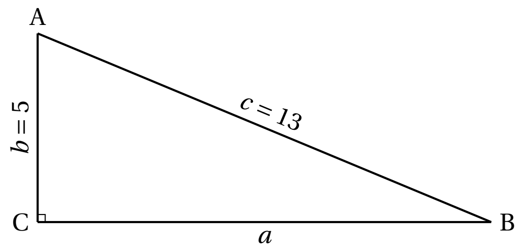
ABC is rectangle. Hence, we have :

$$a^2 + b^2 = c^2$$

or, $a = 3.7, b = 1.4$:

$$\begin{aligned} c^2 &= 3.7^2 + 1.4^2 \\ &= 13.69 + 1.96 \\ &= 15.65 \end{aligned}$$

and hence, taking our calculator: $c = \sqrt{15.65} \approx 3.956$.
Here is a new triangle. Now, we want to know what is a!



ABC is rectangle. Hence, we have :

$$a^2 + b^2 = c^2$$

But we want to calculate a, so we rearrange the terms:

$$a^2 = c^2 - b^2$$

Or, $b = 5, c = 13$:

$$\begin{aligned} a^2 &= 13^2 - 5^2 \\ &= 169 - 25 \\ &= 144 \end{aligned}$$

and hence, taking our calculator (or doing it by hand!): $a = \sqrt{144} = 12$.

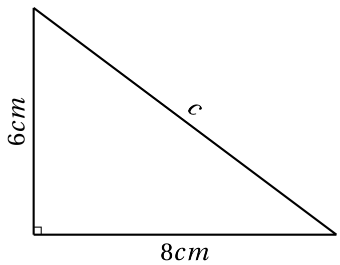
Tip

It is important to note that the 25 is subtracted from the 169 not added.
If it were added the length of side a would work out at 13.928 ($13.928 \approx \sqrt{169+25} = \sqrt{194}$) which is greater than the hypotenuse $c = 13$. This is not possible: the hypotenuse is always the longest side.

Exercise 1.

Find the unknown side in the following, give your answers to 3 significant figures where appropriate.

1.1



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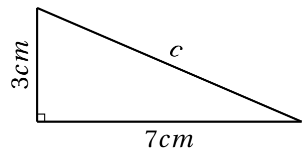

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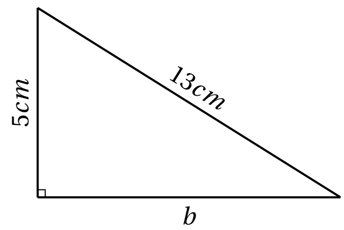

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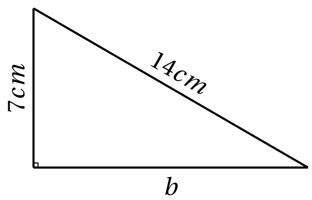

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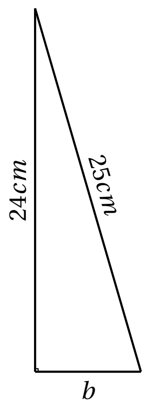

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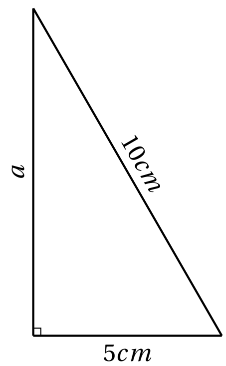


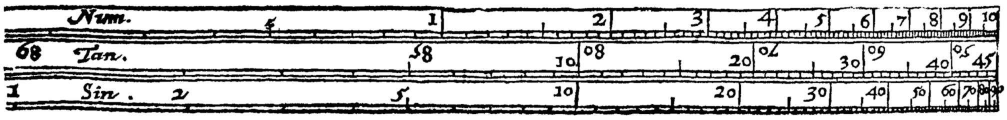
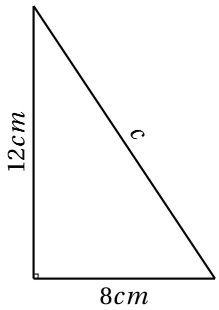


Figure 4: First appearance of sin and tan, Edmond Gunter, 1624.

1.7



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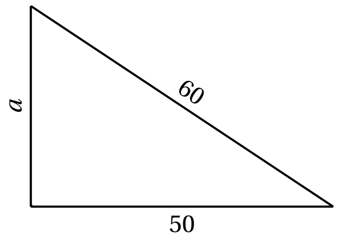

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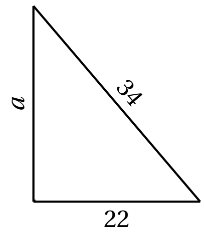

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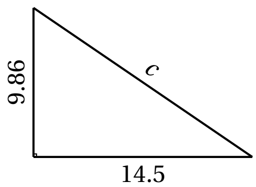

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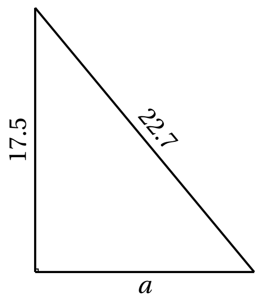
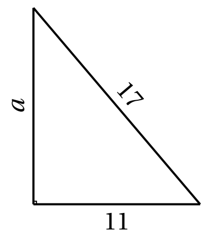


1.12



III Trigonometry

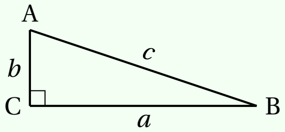
Trigonometry is used to find unknown angles and sides in a right angled triangle. The use of trigonometric functions actually arises from the early connection between mathematics and astronomy. Early work with spherical triangles was as important as plane triangles.

III a) Trigonometric functions

From the right triangle, we can define trigonometric functions.

Def.

TRIGONOMETRIC FUNCTIONS: Trigonometric functions are functions that **depend** on an angle.



The following ratios should be committed to memory although in most examinations they should appear in the list of formulae:

- $\sin\widehat{A} = \frac{a}{c} = \frac{\text{opposite}}{\text{hypothenuse}}$
- $\cos\widehat{A} = \frac{b}{c} = \frac{\text{adjacent}}{\text{hypothenuse}}$
- $\tan\widehat{A} = \frac{a}{b} = \frac{\text{adjacent}}{\text{hypothenuse}} = \frac{\sin\widehat{A}}{\cos\widehat{A}}$

Note that Sin is short for Sine, Cos is short for Cosine and Tan is short for Tangent

Tip

They can be abbreviated to

- $s = \frac{o}{h}$
- $c = \frac{a}{h}$
- $s = \frac{o}{a}$

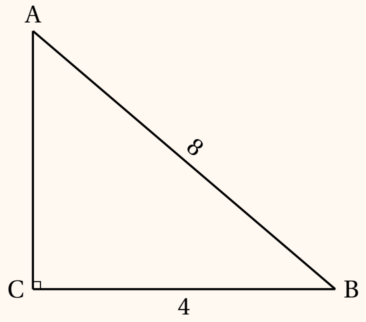
it can be remembered by the word sohcahtoa. If it is too weird, some remember a mnemonic sentence such as "Some Old Houses Creak And Howl Through Old Age".
My favorite solution to remember which side is on the denominator is

- to remember that $sin(0) = 0$.
- we want to pick a side so $\frac{\text{side}}{\text{hypothenuse}} = 0$
- the angle is zero if and only if the opposite side is zero
- $\rightsquigarrow$ the opposite side has to be on the denominator

These ratios allow both to

- define the trigonometric functions such as Sine
- calculate the angle

Main Example



Find the angle $\widehat{A}$.
For that, we must decide which one of the 3 ratios to use: Sine, Cosine or Tangent.
In relation to A, the 4 is the opposite side and the 8 is the hypotenuse.

⇝ Opposite and hypotenuse mean we use sin.

$$\sin\widehat{A} = \frac{\text{opposite}}{\text{hypothenuse}} = \frac{4}{8} = 0.5$$

So we know that $\sin\widehat{A} = 0.5$. How can we calculate $\widehat{A}$?
For that, we will need the **calculator**.
To find the value of $\widehat{A}$, we take the inverse sine using the following keys on your calculator.

- ➔ the Shift (or inverse) key
- ➔ the "Sin" key

So

$$\widehat{A} = \text{Inv Sin}(0.5) = 30^o$$

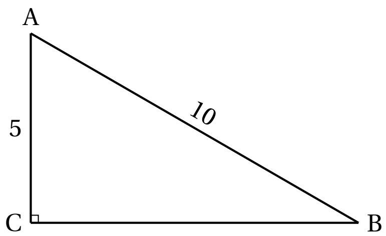
Note that we are using degrees here, so your calculator should be in degree mode not radian mode. If is it in radian mode, your answer will be $\widehat{A} = 0.5235987$.
It is actully equal to $\frac{\pi}{6}$.

Tip

Calculation here also depends on the way your calculator works.

- ➔ some (older type) require the 0.5 to be put in first
- ➔ some (newer type) require the Inv Sin to be put in first

With the newer type the Inv Sin usually appears on your screen.



Let's find the angle $\widehat{A}$.
In relation to A, the 5 is the adjacent side and the 10 is the hypotenuse.

⇝ adjacent and hypotenuse mean we use cos.

$$\cos\widehat{A} = \frac{\text{adjacent}}{\text{hypothenuse}} = \frac{5}{10} = 0.5$$

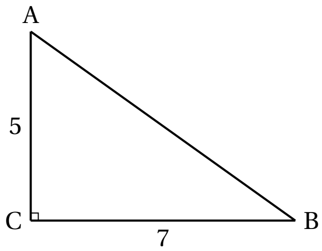
So we know that $\cos\widehat{A} = 0.5$.
To find the value of $\widehat{A}$, we take the inverse cosine using the following keys on your calculator.

➔ the Shift (or inverse) key

➔ the "Cos" key

So

$$\widehat{A} = \text{Inv Cos}(0.5) = 60^o$$



Let's find the angle $\widehat{B}$.
In relation to B, the 5 is the adjacent side and the 7 is the opposite side.

⇝ adjacent and opposite mean we use tan.

$$\begin{aligned}\tan\widehat{B} &= \frac{\text{adjacent}}{\text{opposite}}\\ &= \frac{5}{7}\\ &= 0.7142857143\end{aligned}$$

You should not round this figure

So we know that $\tan\widehat{B} = 0.7142857143$.
To find the value of $\widehat{B}$, we take the inverse tangeant using the following keys on your calculator.

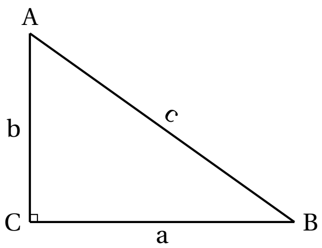
➔ the Shift (or inverse) key

➔ the "Tan" key

So

$$\widehat{B} = \text{Inv Tan}(0.7142857143) = 35.54^o$$

Exercise 2.

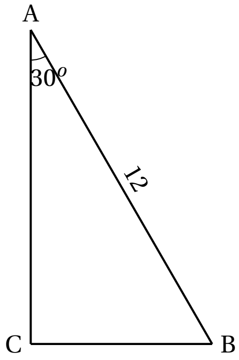


In the triangle ABC above, calculate angle $\widehat{ABC}$. Give your answers to 3 significant figures where appropriate.
The known sides are:

2.1 b=4 and c=8
2.2 b=4 and a=5
2.3 a=2 and c=3.1
2.4 b=3 and c=6.708
2.5 b=7.1 and a=9.7
2.6 a=0.5 and c=1.4
2.7 b=2.65 and a=2.15
2.8 b=14.9 and a=2.4
2.9 a=3 and c=6
2.10 b=2.5 and a=6

IV Finding the unknown side

One of the use of knowing an angle is that it can help us to identify the length of a side in a right triangle.



Can we calculate AC ?

With all trigonometry problems, the first task is to find out which trigonometric ratio to use from the given information. we have or want:

- ➔ the angle ($\widehat{A} = 30^0$)
- ➔ the hypothenuse ($AB = 12$)
- ➔ the adjacent side (AC)

Adjacent and hypotenuse means we should use Cos:

$$\cos \widehat{A} = \frac{adjacent}{hypothenuse} = \frac{AC}{AB}$$

It means that, multiplying both sides with AB:

$$AC = \cos \widehat{A} \times AB$$

and

$$\begin{aligned} AC &= \cos \widehat{A} \times AB \\ &= \cos(30^o) \times 12 \\ &= 0.866 \times 12 \\ &= 10.39 \end{aligned}$$

If we want to calculate BC: we have or want:

- ➔ the angle ($\widehat{A} = 30^0$)
- ➔ the hypothenuse ($AB = 12$)
- ➔ the oposite side (BC)

opposite and hypotenuse means we should use Sin:

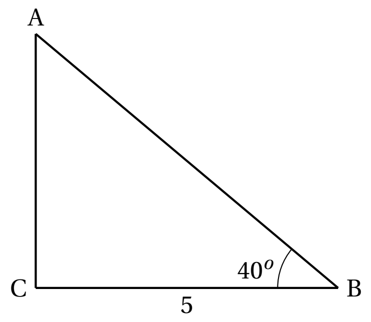
$$\sin \widehat{A} = \frac{opposite}{hypothenuse} = \frac{BC}{AB}$$

It means that, multiplying both sides by AB:

$$BC = \sin \widehat{A} \times AB$$

and

$$\begin{aligned} AC &= \sin \widehat{A} \times AB \\ &= \sin(30^o) \times 12 \\ &= 0.5 \times 12 \\ &= 6 \end{aligned}$$



Can we calculate AC ? we have or want:

➔ the angle ($\widehat{B} = 30^0$)

➔ the adjacent side (BC = 5)

➔ the opposite side (AC)

Adjacent and opposite means we should use the tangeant:

$$\tan\widehat{B} = \frac{opposite}{adjacent} = \frac{AC}{BC}$$

It means that, multiplying both side by BC:

$$AC = \tan\widehat{B} \times BC$$

and

$$\begin{aligned} AC &= \tan\widehat{B} \times BC \\ &= \tan(40^o) \times 5 \\ &= 0.839 \times 5 \\ &= 4.20 \end{aligned}$$

Can we calculate AB ? we have or want:

➔ the angle ($\widehat{B} = 30^0$)

➔ the adjacent side (BC = 5)

➔ the hypothenuse (AB)

Adjacent and hypothenuse means we should use the cosine:

$$\cos\widehat{B} = \frac{adjacent}{hypotenuse} = \frac{BC}{AB}$$

It means that, multiplying both sides with AB:

$$AB\cos\widehat{B} = BC$$

and, if we divide both sides by $\cos\widehat{B}$ (which is not 0!):

$$AB = \frac{BC}{\cos\widehat{B}}$$

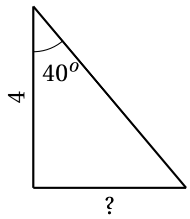
and

$$\begin{aligned} AB &= BC/\cos\widehat{B} \\ &= 5/\cos(40^o) \\ &= 5/0.766 \\ &= 6.53 \end{aligned}$$

Exercise 3.

Find the sides marked ?, giving your answers to 3 significant figures where appropriate.

3.1



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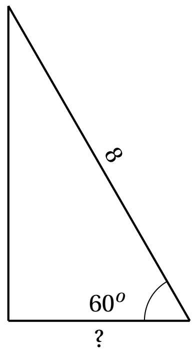

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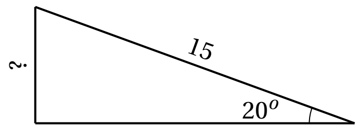

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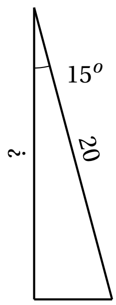

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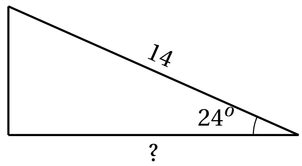

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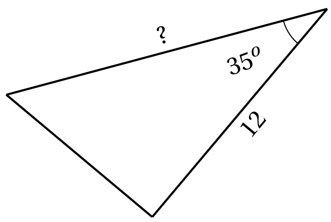

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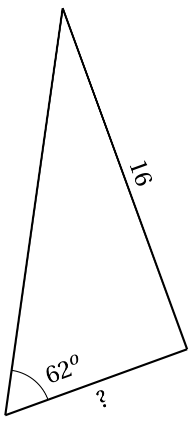

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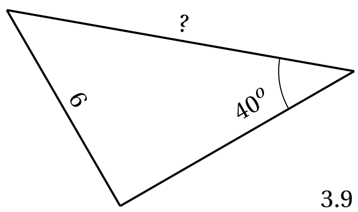

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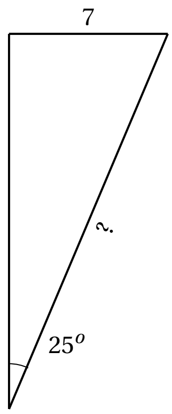

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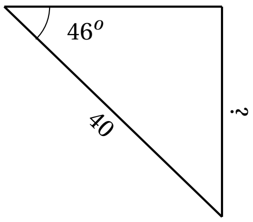

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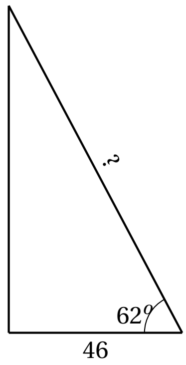

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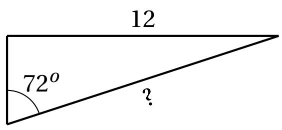
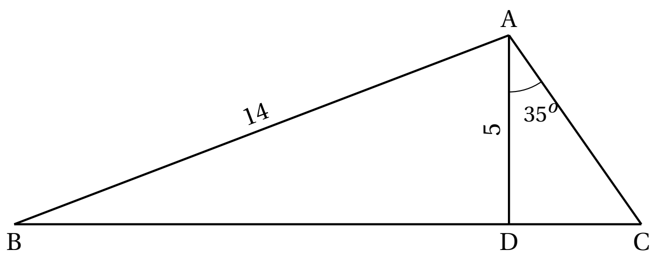


3.12

V Practical problems

Find BC and angle $\widehat{BAD}$, giving your final answers to 2 decimal places:



Tip

With this problem, as with many others, there is not a specific method that you can follow. You hence have to:

1. the lengths and/or angle that you need to find the solution.
   This is the hardest part, but it is essential in Engineering: find a solution and identify what you need to complete it.
2. compute these lengths/angle by using the adequate formula.
   Sometimes, several formulas can work ! Try to find the easiest one !

For calculating BC, we must find BD and DC separately. Adding them together will give BC.

These can be found in many different ways, all of which are correct. Nevertheless, usually, one method is easier.

For example we could find BD using these two different methods

1. (a) using AB and AD and the cosine ratio:

$$\begin{aligned}\cos\widehat{BAD} &= \frac{AD}{AB}\\ &= 5/14\end{aligned}$$

   (b) using the inverse cosine to get

$$\begin{aligned}\widehat{BAD} &= InvCos\left(\tfrac{5}{14}\right)\\ &= 69.0752^o\end{aligned}$$

   Note that we have $\widehat{BAD}$!

   (c) using the sine ratio:

$$\sin\widehat{BAD} = \frac{BD}{AB}$$

   to get

$$\begin{aligned} BD &= AB \times \sin \widehat{BAD} \\ &= 14 \times \sin(69.0752) \\ &= 13.0767 \end{aligned}$$

Note that the tan ratio could have been used !

2. using Pythagoras theorem:
   $AB^2 = AD^2 + BD^2$
   $$\Rightarrow \begin{aligned} BD &= \sqrt{AB^2 - AD^2} \\ &= 13.0767 \end{aligned}$$

The second one seems easier. But the first one also give us $\widehat{BAD}$!

Then, a way of calculating DC can be:

1. using the tan ratio: $\tan \widehat{DAC} = \dfrac{DC}{AD}$

2. leads to $DC = AD \times \tan \widehat{DAC} = 5 \times \tan(35^0) = 3.5010$

It means that

$$\begin{aligned} BC &= BD+DC \\ &= 13.0767 + 3.5010 \\ &= 16.5777 \end{aligned}$$

and hence

$$BC = 16.58$$

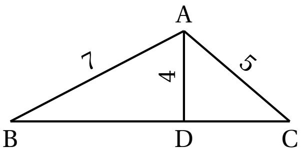
And, having already $\widehat{BAD} = 69.0752^o$:

$$\widehat{BAD} = 69.08^o$$

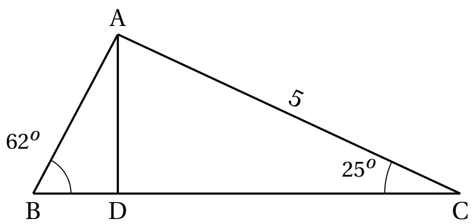
Tip

It is important to not round off any intermediate calculations.
You can not quote your final answers to 4 decimal places if you have only worked to 2 decimal places throughout. Work to more decimal places than you need or use the full display of your calculator each time.

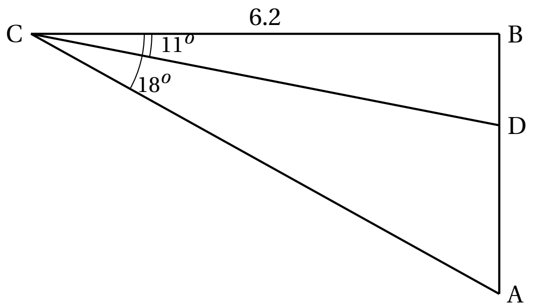
Exercise 4.

4.1 

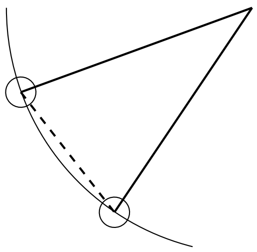
Calculate $\widehat{BAC}$ and BC.

4.2 

Calculate BD, AD, AB and DC

4.3 

Angle $\widehat{DCB} = 11^o$ and angle $\widehat{ACD} = 18^o$. Calculate BD, AD, AB and DC.

4.4 

The figure shows two adjacent holes of a set of ten holes equi-spaced on a pitch circle diameter of $65mm$. Calculate the centre distance between two adjacent holes.

VI Solutions to exercises

Solution 1.

1.1	c=10cm	1.2	c=7.62cm	1.3	b=12cm	1.4	b=12.1cm	1.5	b=12cm
1.6	a=8.66cm	1.7	c=14.4cm	1.8	a=33.2	1.9	a=25.9	1.10	c=17.5
1.11	a=14.5	1.12	a=13.0						

Solution 2.

2.1	$\widehat{ABC} = 30^o$	2.2	$\widehat{ABC} = 38.7^o$	2.3	$\widehat{ABC} = 49.8^o$	2.4	$\widehat{ABC} = 24.1^o$	2.5	$\widehat{ABC} = 36.2^o$
2.6	$\widehat{ABC} = 69.1^o$	2.7	$\widehat{ABC} = 50.9^o$	2.8	$\widehat{ABC} = 80.8^o$	2.9	$\widehat{ABC} = 60^o$	2.10	$\widehat{ABC} = 22.6^o$

Solution 3.

3.1	3.36	3.2	4	3.3	5.13	3.4	19.3
3.5	12.8	3.6	14.6	3.7	8.51	3.8	9.33
3.9	16.6	3.10	28.8	3.11	98.0	3.12	12.6

Solution 4.

4.1

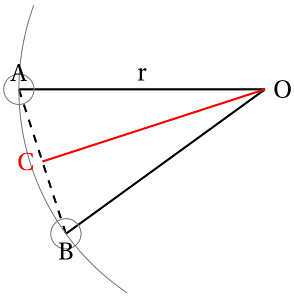
- $\widehat{BAC} = \widehat{BAD} + \widehat{DAC}$.
  - $\cos\widehat{BAD} = AC/AB \Rightarrow \widehat{BAD} = InvCos(AC/AB) = 55.1501^o$
  - $\cos\widehat{DAC} = DA/AC \Rightarrow \widehat{DAC} = InvCos(DA/AC) = 36.8699^o$
  - $\widehat{BAC} = 92.02^o$
- $BC = BD = DC$
  - $AB^2 = BD^2 + AD^2 \Rightarrow BD = \sqrt{AB^2 - AD^2} = 5.7446$
  - $AB^2 = BD^2 + AD^2 \Rightarrow BD = \sqrt{AB^2 - AD^2} = 3$
  - $BC = 8.75$

4.2

- $\sin\widehat{DCA} = AD/AC \Rightarrow AD = \sin\widehat{DCA} \times AC = 2.11$.
- $\tan\widehat{BDA} = AD/BD \Rightarrow BD = AD/\tan\widehat{BDA} = 1.12$.
- $AB^2 = BD^2 + AD^2 \Rightarrow AB = \sqrt{BD^2 + AD^2} = 2.39$
- $\cos\widehat{DCA} = DC/AC \Rightarrow DC = \cos\widehat{DCA} \times AC = 4.53$.

4.3

- $\tan\widehat{DCB} = BD/CB \Rightarrow BD = \tan\widehat{DCB} \times CB = 1.21$.
- For AB
  - $\widehat{ACB}$, which is $\widehat{ACB} = \widehat{ACD} + \widehat{DCB} = 29^o$
  - $\tan\widehat{ACB} = AB/CB \Rightarrow AB = \tan\widehat{ACB} \times CB = 3.44$.
- $DA = AB - BD = 2.23$
- $\cos\widehat{DCB} = BD/CD \Rightarrow CD = BD/\cos\widehat{DCB} = 6.32$.



4.4

Let's define r as the radius (half the diameter), and $a = \widehat{AOB}$ the angle between the holes and the center. Say C is between the two holes. To solve the problem, one strategy is to identify half the distance $d = AC = CB$:

- The angle a between holes and center is $a = 360/10$
- now, we have $\sin\widehat{AOC} = \sin(a/2) = d/r \Rightarrow d = r \times \sin(a/2)$
- It means that $AC = 2d = 2r\sin(a/2) = 20.09mm$

Bibliography

[Croft and Davidson, 2016] Croft, A. and Davidson, R. (2016). *Foundation Maths.* Pearson.