Mathematics for Engineers–ENG 3009, 2018-2019

Simultaneous equations in 2 and 3 unknowns

FLORIMOND GUENIAT & VIJAY VENKATESH
WITH BRIAN SMITH

BIRMINGHAM CITY
University

In this chapter, we will look at solving several equations at the same time.

I Introduction

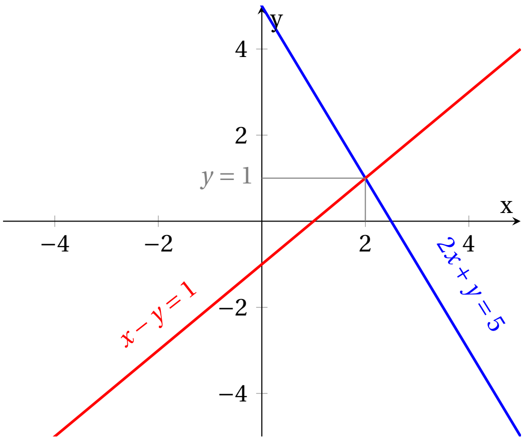
Often, we want to solve a problem with two unknowns (e.g., the temperature and time of the day) ato two constraints (for example the collision between two bodies in which both momentum and kinetic energy are conserved).

Def.

SIMULTANEOUS EQUATIONS: The term simultaneous equations refers to

1. a collection of equations
2. that are all true at the same time
3. for the same values or parameters

If we are given the equation $2x + y = 5$ then it has an infinite number of solutions. But if we are also given the equation $xy = 1$ and asked to find values of x and y that satisfy both the equations then this is a set of simultaneous equations.



When plotting both equations, there is only one point at the intersection, meaning one solution.

Main Example

For instance, Paul has been two times to the grocery shop. He bought the first time one apple and one pear for three pounds.
Can we deduce the price of an apple and a pear ? So far, a pear might be 1.5$ and an apple might be 1.5$. But it might be as well 1.75$ for an apple and 1.25$ for a pear. We can not deduce the price yet, we have not enough equations.
The second time, enjoying the pear more than the apple, Paul bought one apple, and two pears, for a total of five pounds.
We can then deduce a system of equations, when x is the price of an apple, and y the price of a pear.

$$\mathscr{S} : \begin{cases} x + \quad y & = 3 \\ x + \quad 2y & = 5 \end{cases}$$

Solving this system will allow us to deduce the prices of apples and pears.

There are a number of ways of solving these. This chapter will consider the easier non- matrix methods, such as substitution, elimination and graphical.

Tip

The solution of simultaneous equations is *all* the unknowns. For instance, having the system

$$\mathscr{S}:\begin{cases} x+ & y & =3 \\ x+ & 2y & =5 \end{cases}$$

then the solution is $x = 1$ and $y = 2$. You cannot say that $x = 1$ is a solution and $y = 2$ is another solution.

II Algebraic methods for solving simultaneous equations with 2 unknowns

In this section, we aim at solving simultaneous equations using algebraic methods.

Def.

ALGEBRAIC METHODS: Algebraic methods aims at reducing the number of unknowns in a system by *manipulating the equations.*
For instance, we start from a system with 2 equations and 2 unknowns

$$\mathscr{S}:\begin{cases} ax+ & by & =e \\ cx+ & dy & =f \end{cases}$$

, and identify a sub-system of 1 equation with 1 unknown, say:

$$x = g$$

x is hence known, it is a part of the solution.
This knowledge is used in the original system to identify the second unknown, for instance:

$$by = f - ag \Rightarrow y = \frac{f - ag}{b}$$

Both unknwons have hence be calculated, and it form the solution.
There are two algebraic methods, the Substitution Method and the Elimination Method.

II a) Substitution

Def.

SUBSTITUTION: The substitution method involves rearranging one equation, to make one of the variables the subject. Then, this variable is substituted in the other equation. It allows to solve it for only one variable.

In other words, find the value of y in terms of x (or vice versa) for one of the two equations, and then substitute this value into the other equation.

Main Example

Let's consider the system $\mathscr{S}$:

$$\mathscr{S}:\begin{cases} 2x+ & y & =5 & \text{eq. 1} \\ x- & y & =1 & \text{eq. 2} \end{cases}$$

We start by rearranging eq. 2 with x as the subject:

$$x-y=1 \Leftrightarrow x=1+y$$

We substitute x with $1+y$ in eq. 2:

$$\begin{aligned} 2x+y=5 &\Rightarrow 2(1+y)+y=5 && \text{substitution } x\leftarrow 1+y \\ &\Rightarrow 2+2y+y=5 && \text{removing brackets} \\ &\Rightarrow 2+3y=5 && \text{rearranging the ys} \\ &\Rightarrow 3y=3 && \text{y as the subject} \\ &\Rightarrow y=1 \end{aligned}$$

y is now known.
To find the value of x, we substitute back the known value of y into either of the two equations.
It does not matter which one because we should get the same answer. If you get a different answer from the two equations then the value found for y is incorrect, it can be useful to find errors !
Using eq. 2:

$$\begin{aligned} 2x+y=5 &\Rightarrow 2x+1=5 && \text{substitution } y\leftarrow 1 \\ &\Rightarrow 2x=4 && \text{x as the subject} \\ &\Rightarrow x=2 \end{aligned}$$

We could have use eq. 1:

$$\begin{aligned} x-y=1 &\Rightarrow x-1=1 && \text{substitution } y\leftarrow 1 \\ &\Rightarrow x=2 && \text{x as the subject} \end{aligned}$$

The result is the same. The solution of the system $\mathscr{S}$ is $x=2$ and $y=1$.
As a check the values we have found should satisfy both equations:

- ➔ eq. 1: $2x+y=5 \Rightarrow 2\times 2+1\times 1=5$ as required.
- ➔ eq. 2: $x-y=1 \Rightarrow 1\times 21\times 1=1$ as required.

Generally, it does not matter which variable is chosen or which equation is chosen although some choices will be easier than others.

Let's consider the system $\mathscr{S}_\infty$:

$$\mathscr{S}_\infty:\begin{cases} x+ & y & =7 & \text{eq. 1} \\ 3x+ & 2y & =17 & \text{eq. 2} \end{cases}$$

There is several paths to solve $\mathscr{S}_\infty$:

- ➔ Let's start by the following path. We will rearrange equation 1 for y and substitute this into equation 2.

  First, let's rearrange eq.1.

$$x+y=7 \Leftrightarrow y=7-x$$

We substitute y in eq. 2:

$$\begin{aligned} 3x+2y=17 \quad &\Rightarrow \quad 3x+2\times(7-x)=17 && \text{substitution } y\leftarrow 7-x \\ &\Rightarrow \quad 3x+14-2x=17 && \text{removing brackets} \\ &\Rightarrow \quad x+14=17 && \text{rearranging the xs} \\ &\Rightarrow \quad x=3 \end{aligned}$$

To find the value of y, we substitute the known value of x into either of the two equations. Once again it does not matter which one, because we should get the same answer! Using eq. 1:

$$\begin{aligned} x+y=7 \quad &\Rightarrow \quad 3+y=7 && \text{substitution } x\leftarrow 3 \\ &\Rightarrow \quad y=4 \end{aligned}$$

As a check the values we have found, $x=3$, $y=4$ should satisfy both equations:

- eq. 1: $x+y=7 \Rightarrow 3+4=7$ as required.
- eq. 2: $3x+2y=17 \Rightarrow 3\times 3+2\times 4=17$ as required.

➔ Instead, let's now rearrange equation 2 for y and substitute this into equation 1!

First, let's rearrange eq.2.

$$3x+2y=17 \Leftrightarrow y=\frac{17-3x}{2}$$

This is why some choices can be easier.

We substitute y in eq. 1:

$$\begin{aligned} x+y=7 \quad &\Rightarrow \quad x+\left(\frac{17-3x}{2}\right)=7 && \text{substitution } y\leftarrow \frac{17-3x}{2} \\ &\Rightarrow \quad x+\frac{17}{2}-\frac{3x}{2}=7 && \text{removing brackets} \\ &\Rightarrow \quad -\frac{x}{2}+\frac{17}{2}=7 && \text{rearranging the xs} \\ &\Rightarrow \quad \frac{x}{2}=\frac{17}{2}-7 && \text{x as the subject} \\ &\Rightarrow \quad x=3 \end{aligned}$$

To find the value of y, we substitute the known value of x into either of the two equations. Once again it does not matter which one, because we should get the same answer! Using eq. 1:

$$\begin{aligned} x+y=7 \quad &\Rightarrow \quad 3+y=7 && \text{substitution } x\leftarrow 3 \\ &\Rightarrow \quad y=4 \end{aligned}$$

and we have the same results as before !

Exercise 1.

Using the method of substitution, solve the following:

1.1

$$\mathscr{S}_1 : \begin{cases} x+ \ \ y &= 6 \\ 2x+ \ 3y &= 14 \end{cases}$$

1.2

$$\mathscr{S}_2 : \begin{cases} 2x- \ \ y &= 11 \\ x+ \ 2y &= -7 \end{cases}$$

1.3

$$\mathscr{S}_3 : \begin{cases} 3x- \ 4y &= 7 \\ x- \ 2y &= 5 \end{cases}$$

1.4

$$\mathscr{S}_4 : \begin{cases} 3x+ \ 2y &= 8 \\ 2x- \ \ y &= 3 \end{cases}$$

1.5

$$\mathscr{S}_5 : \begin{cases} 4x- \ 5y &= 3 \\ 7x- \ 10y &= 5 \end{cases}$$

1.6

$$\mathscr{S}_6 : \begin{cases} 4x- \ 3y &= 6 \\ 2x- \ \ y &= 4 \end{cases}$$

II b) Elimination

The method of elimination is based on manipulating together equations. Let's first see how to add or subtract equations.

Prop.

MANIPULATING EQUATIONS TOGETHER: Equations can be added (or subtracted) to make new equations.

- ➔ we add the right hand side of the eq. 1 to the right hand side of eq.2. That forms the right hand side of the new equation.
- ➔ and then we add the left hand side of the eq. 1 to the left hand side of eq.2. That forms the left hand side of the new equation.

In particular, if we have $a = b$ and $c = d$, then the following are true:

- ➔ $a + c = b + d$
  adding eq.1 and 2.
- ➔ $a + nc = b + nd$
  adding eq.1 and n times eq.2.
- ➔ $a + d = b + c$
  adding eq1. and 2.
- ➔ $a - c = b - d$
  subtracting eq.2 from eq.1
- ➔ $a - d = b - c$
  subtracting eq.2 from eq.1
- ➔ $a - nc = b - nd$
  subtracting n times eq.2 from eq.1

As we are adding on both sides of an equation the *same* quantity, the property is hence true. Et voilà !

We can now define the method of elimination.

Def.

ELIMINATION: The elimination method involves eliminating one of the variables by adding or subtracting multiples of one equation to another. Then, the remaining variable is substituted in the other equation.

In other words, make the coefficients of one of the variables the same value in both equations. Then either add the equations or subtract one equation from the other (whichever is appropriate) to form a new equation, that only contains one variable, say x. This is referred to as eliminating the variable.
Solve the equation thus obtained gives x. Then substitute the value found for the variable in one of the given equations and solve it for the other variable, say y.

As before, it does not matter which variable is chosen although again some choices will be easier than others and some sets of equations lend themselves to a particular method as the examples will show.

Main Example

Let's consider the following system:

$$\mathscr{S}: \begin{cases} 2x+ \quad y \quad = 5 & \text{eq. 1} \\ x- \quad y \quad = 1 & \text{eq. 2} \end{cases}$$

There is a $+y$ in equation 1 and a y in equation 2, so y will disappear if we add the two equations together. Practically, adding eq.1 and eq. 2 is done as:

- ➔ Right hand side:

$$\underbrace{2x+y}_{\text{eq. 1}}+\underbrace{x-y}_{\text{eq.2}}=3x$$

- ➔ Left hand side:

$$\underbrace{5}_{\text{eq. 1}}+\underbrace{1}_{\text{eq.2}}=6$$

and hence, we have

$$3x=6 \Leftrightarrow x=2$$

To find the value of y, we substitute the known value of x into either of the two equations. Again it does not matter which one because (we should get the same answer).
Using eq. 2:

$$\begin{aligned} x-y=1 \quad &\Rightarrow \quad (2)-y=1 && \text{substitution } x \leftarrow 2 \\ &\Rightarrow \quad 2-y=1 && \text{rearranging the ys} \\ &\Rightarrow \quad -y=-1 && \text{y as the subject} \\ &\Rightarrow \quad y=1 \end{aligned}$$

As a check the values we have found, $x=2, y=1$ should satisfy both equations:

- ➔ eq. 1: $2x+y=5 \Rightarrow 2\times 2+1=5$ as required.
- ➔ eq. 2: $x-y=1 \Rightarrow 2-1=1$ as required.

Let's now consider the following system:

$$\mathscr{S}_\infty: \begin{cases} 2x+ \quad y \quad = 11 & \text{eq. 1} \\ x+ \quad y \quad = 7 & \text{eq. 2} \end{cases}$$

There is a $+y$ in equation 1 and a $+y$ in equation 2, so y will disappear if we subtract the two equations from each other. Practically, subtracting eq.2 from eq. 1 is done as:

- ➔ Right hand side:

$$\underbrace{2x+y}_{\text{eq. 1}}-\underbrace{(x+y)}_{\text{eq.2}}=x$$

- ➔ Left hand side:

$$\underbrace{11}_{\text{eq. 1}}-\underbrace{7}_{\text{eq.2}}=4$$

and hence, we have $x=4$.

To find the value of y, we substitute the known value of x into either of the two equations. Again it does not matter which one because (we should get the same answer).

Using eq. 2:

$$\begin{aligned} x+y=7 \quad &\Rightarrow \quad (4)+y=7 && \text{substitution } x \leftarrow 4 \\ &\Rightarrow \quad y=7-4 && \text{y as the subject} \\ &\Rightarrow \quad y=3 \end{aligned}$$

As a check the values we have found, $x=4$, $y=3$ should satisfy both equations:

- ➔ eq. 1: $2x+y=11 \Rightarrow 2\times 4+3=11$ as required.
- ➔ eq. 2: $x+y=7 \Rightarrow 4+3=7$ as required.

Tip

Note that, from the examples, it can be seen that to eliminate the variable

- ➔ the equations are added if the signs are different
- ➔ the equations are subtracted if the signs are the same

Let's now consider the following system:

$$\mathscr{S}_\epsilon : \begin{cases} 3x+ \quad 2y \quad =17 & \text{eq. 1} \\ \;\; x+ \quad\;\; y \quad =7 & \text{eq. 2} \end{cases}$$

This set of equations need "adjusting" before we can eliminate one of the variables. There is a few ways to do so:

- ➔ One option is to multiply equation 2 by 2. This will give a $2y$ in that equation as well, then we can subtract the equations. Practically, subtracting 2 times eq.2 from eq. 1 is done as:
  - Right hand side:

$$\underbrace{3x+2y}_{\text{eq. 1}}-\underbrace{2\times(x+y)}_{\text{2 times eq.2}}=x$$

  - Left hand side:

$$\underbrace{17}_{\text{eq. 1}}-\underbrace{2\times 7}_{\text{2 times eq.2}}=3$$

  and hence, we have $x=3$.

  To find the value of y, we substitute the known value of x into either of the two equations. Again it does not matter which one because (we should get the same answer).

  Using eq. 2:

$$\begin{aligned} x+y=7 \quad &\Rightarrow \quad (3)+y=7 && \text{substitution } x \leftarrow 3 \\ &\Rightarrow \quad y=7-3 && \text{y as the subject} \\ &\Rightarrow \quad y=4 \end{aligned}$$

  As a check the values we have found, $x=3$, $y=4$ should satisfy both equations:

  - eq. 1: $3x+2y=17 \Rightarrow 3\times 3+2\times 4=17$ as required.
  - eq. 2: $x+y=7 \Rightarrow 3+4=7$ as required.

- ➔ We could have just as easily multiplied equation 2 by 3 and proceeded as follows, this would have given $3x$ in both equations and subtracting would have found the value of y. Practically, subtracting three times eq.2 from eq. 1 is done as:

- Right hand side:

$$\underbrace{3x+2y}_{\text{eq. 1}} - \underbrace{3\times(x+y)}_{\text{3 times eq.2}} = -y$$

- Left hand side:

$$\underbrace{17}_{\text{eq. 1}} - \underbrace{2\times 7}_{\text{3 times eq.2}} = -4$$

and hence, we have $-y=-4 \Leftrightarrow y=4$.

To find the value of x, we substitute the known value of y into either of the two equations. Again it does not matter which one because (we should get the same answer).

Using eq. 2:

$$\begin{aligned} x+y=7 \quad &\Rightarrow \quad x+(4)=7 \quad && \text{substitution } y\leftarrow 4 \\ &\Rightarrow \quad x=7-4 && \text{x as the subject} \\ &\Rightarrow \quad x=3 \end{aligned}$$

Exercise 2.

Using the method of substitution, solve the following:

2.1

$$\mathscr{S}_1 : \begin{cases} x+ & 2y &= 9 \\ x- & 2y &= 5 \end{cases}$$

2.2

$$\mathscr{S}_2 : \begin{cases} 3x- & 5y &= 4 \\ 3x- & 3y &= 6 \end{cases}$$

2.3

$$\mathscr{S}_3 : \begin{cases} 2x- & 5y &= 2 \\ x+ & 5y &= 16 \end{cases}$$

2.4

$$\mathscr{S}_4 : \begin{cases} 4x- & y &= 4 \\ 2x- & y &= 0 \end{cases}$$

2.5

$$\mathscr{S}_5 : \begin{cases} 5x- & 2y &= 26 \\ 2x- & 2y &= 14 \end{cases}$$

2.6

$$\mathscr{S}_6 : \begin{cases} 5x+ & 3y &= 19.5 \\ 6x- & y &= 13.5 \end{cases}$$

2.7

$$\mathscr{S}_7 : \begin{cases} 3x+ & 2y &= 8 \\ 2x- & y &= 3 \end{cases}$$

2.8

$$\mathscr{S}_8 : \begin{cases} 4x- & 5y &= 3 \\ 7x- & 10y &= 5 \end{cases}$$

2.9

$$\mathscr{S}_9 : \begin{cases} 4x- & 3y &= 6 \\ 2x- & y &= 4 \end{cases}$$

2.10

$$\mathscr{S}_{10} : \begin{cases} x- & 3y &= -4 \\ 3x+ & y &= 8 \end{cases}$$

2.11

$$\mathscr{S}_{11} : \begin{cases} 4x- & y &= 6 \\ 3x- & 0.5y &= 6 \end{cases}$$

2.12

$$\mathscr{S}_{12} : \begin{cases} 4x+ & 3y &= 44 \\ 2x- & 5y &= 9 \end{cases}$$

III Graphical method for solving simultaneous equations of 2 unknowns

III a) Method

This method involves plotting graphs of the two equations on the same axis and finding out where they cross or intersect. This point is the solution of the simultaneous equations. If you think about the first line then any point on that line satisfies that equation and any point on the second line satisfies the second equation.

It follows therefore that the only point that satisfies both equations is at the intersection point of the two lines.

Main Example

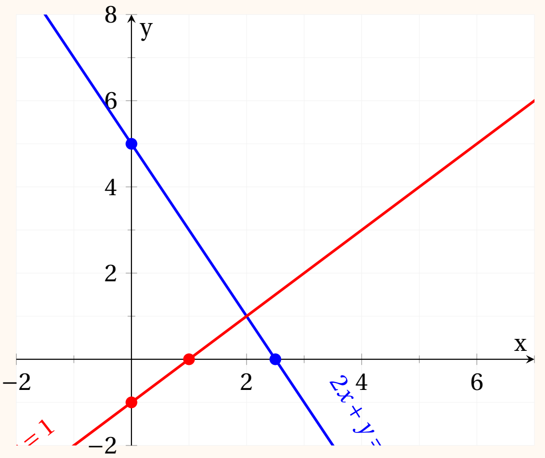
Let's consider the system $\mathscr{S}$:

$$\mathscr{S}:\begin{cases} 2x+ \quad y \quad = 5 & \text{eq. 1} \\ \quad x- \quad y \quad = 1 & \text{eq. 2} \end{cases}$$

We must plot the two equations. There are a number of ways of doing this, we are going to use the method by which you find out where the line crosses the two axes.

- ➔ eq. 1
  - x-axis:
    $2x + y = 5$ crosses the x axis when $y = 0 \Rightarrow 2x = 5 \Rightarrow x = 2.5$. The point is hence (2.5,0).
  - y-axis:
    $2x + y = 5$ crosses the y axis when $x = 0 \Rightarrow y = 5$. The point is hence (0,5).
- ➔ eq. 2
  - x-axis:
    $x - y = 1$ crosses the x axis when $y = 0 \Rightarrow x = 1$. The point is hence (1,0).
  - y-axis:
    $x - y = 1$ crosses the y axis when $x = 0 \Rightarrow -y = 1 \Rightarrow y = -1$. The point is hence (0,-1).

We now plot these points on the same axes, draw the curves. Finding out the intersection point gives us the solution.



The solution is the intersection point of the two lines, $x = 2$ and $y = 1$.

Let's consider the system $\mathscr{S}$:

$$\mathscr{S}:\begin{cases} x- \quad 2y \quad = 2 & \text{eq. 1} \\ x+ \quad y \quad = 5 & \text{eq. 2} \end{cases}$$

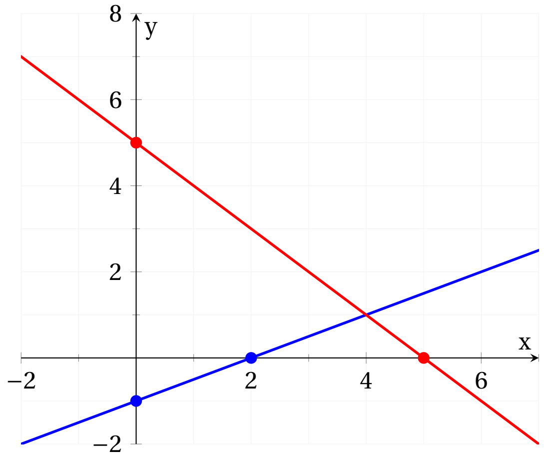
- ➔ eq. 1
  - x-axis:
    $x - 2y = 2$ crosses the x axis when $y = 0 \Rightarrow x = 2$. The point is hence (2,0).

- y-axis:
  $x-2y=2$ crosses the y axis when $x=0 \Rightarrow -2y=2 \Rightarrow y=-1$. The point is hence (0,-1).

➔ eq. 2

- x-axis:
  $x+y=5$ crosses the x axis when $y=0 \Rightarrow x=5$. The point is hence (5,0).
- y-axis:
  $x+y=5$ crosses the y axis when $x=0 \Rightarrow y=5$. The point is hence (0,5).

We now plot these points on the same axes, draw the curves. Finding out the intersection point gives us the solution.



The solution is the intersection point of the two lines, $x=4$ and $y=1$.

Exercise 3.

Solve the following graphically

3.1

$$\mathscr{S}_1 : \begin{cases} 4x - 4y = 4 \\ 2x - y = 0 \end{cases}$$

3.2

$$\mathscr{S}_2 : \begin{cases} x + y = 6 \\ 2x + 3y = 14 \end{cases}$$

3.3

$$\mathscr{S}_3 : \begin{cases} 2x - 5y = 2 \\ x + 5y = 16 \end{cases}$$

Exercise 4.

Solve the following by the method that you consider to be the best.

4.1

$$\mathscr{S}_1 : \begin{cases} 3x - 2y = 11 \\ x + y = 7 \end{cases}$$

4.2

$$\mathscr{S}_2 : \begin{cases} 4x - y = 5 \\ 2x - 2y = -2 \end{cases}$$

4.3

$$\mathscr{S}_3 : \begin{cases} 5x + 2y = -8 \\ 4x - 5y = 6 \end{cases}$$

4.4

$$\mathscr{S}_4 : \begin{cases} 5x- & 2y & = 7 \\ x+ & y & = 3.5 \end{cases}$$

4.5

$$\mathscr{S}_5 : \begin{cases} x+ & 3y & = 21 \\ x- & 5 & = y \end{cases}$$

4.6

$$\mathscr{S}_6 : \begin{cases} 3x & -4y & = 0 \\ x = & 5- & 2y \end{cases}$$

IV harder problems

The following exercise involves simultaneous equations with linear and quadratic. These should be attempted only after you have studied quadratic equations

Exercise 5.

Solve the following linear and quadratic simultaneous equations

5.1

$$\mathscr{S}_1 : \begin{cases} y = & x^2 + x \\ y = & 4 - 2x \end{cases}$$

5.2

$$\mathscr{S}_2 : \begin{cases} y = & x^2 - x \\ y = & 6 - 6x \end{cases}$$

Exercise 6.

Solve the following graphically

$$\mathscr{S} : \begin{cases} y = & x^2 + 2x \\ y = & x + 2 \end{cases}$$

Exercise 7.

Solve the following (harder examples)

7.1

$$\mathscr{S}_1 : \begin{cases} 3x+ & y & = 1 \\ 2x^2- & y^2 & = -2 \end{cases}$$

7.2

$$\mathscr{S}_2 : \begin{cases} 4x^2+ & y^2 & = 61 \\ 2x+ & y & = 1 \end{cases}$$

7.3

$$\mathscr{S}_3 : \begin{cases} x^2+ & 2y^2 & = 3 \\ x- & 3y & = 2 \end{cases}$$

7.4

$$\mathscr{S}_4 : \begin{cases} x+ & y & = 3 \\ x^2- & y^2 & = -3 \end{cases}$$

7.5

$$\mathscr{S}_5 : \begin{cases} 3x+ & 2y & = 6 \\ & xy & = -12 \end{cases}$$

7.6

$$\mathscr{S}_6 : \begin{cases} & xy & = 30 \\ 3x+ & y & = 21 \end{cases}$$

7.7

$$\mathscr{S}_7 : \begin{cases} 4x- & y & = 7 \\ & xy & = 15 \end{cases}$$

7.8

$$\mathscr{S}_8 : \begin{cases} 3x- & 4y & = 2 \\ & xy & = 2 \end{cases}$$

V Problems with 3 unknowns

We will consider here simultaneous equations in 3 unknowns:

$$\mathscr{S}: \begin{cases} a_1 x + & b_1 y + & c_1 z & = d_1 \\ a_2 x + & b_2 y + & c_2 z & = d_2 \\ a_3 x + & b_3 y + & c_3 z & = d_3 \end{cases}$$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3$ are all known, and the unknowns are x, y and z. For instance, we can consider the following system:

$$\mathscr{S}: \begin{cases} x + & 3y - & 2z & = -1 \\ -1x + & 2y + & 2z & = 3 \\ 3x + & 5y - & 3z & = 14 \end{cases}$$

There are a number of ways of solving simultaneous equations in 3 unknowns such as advanced matrix methods as

- ➔ Gaussian Elimination
- ➔ Cramers rule

If powerful, these methods are out of our current scope. We will look at these methods in the chapter *Matrix*.

In this section, we aim at solving simultaneous equations algebraically, as in Sec. II.

Tip

The main idea is hence to start with 3 equations in 3 unknowns, and to reduce this to 2 equations in 2 unknowns. The attentive reader will see that, in Sec. II, we had the same idea: reducing two equations of two unknowns to one with one unknown.
Once the reduction to a system with 2 unknowns is achieved, we can then follow one of the methods discussed earlier in Sec. II.

The mathematical tools have been already given in Sec. II. We will just apply them to larger systems (it is actually possible to do the same for systems of n equations with n unknowns), through a few examples.

The examples cover two types:

- ➔ sets of equations where all 3 equations contain all 3 variables
- ➔ sets of equations where some of the variables do not appear in all of the equations

The first type are usually solved by rearranging one equation with respect to one variable (say, x) and substituting the expression for the variable (x) into both the other two equations. As a result, the two other equations do not exhibit the chosen variable (x), effectively reducing the problem to one of 2 equations in 2 unknowns.

The second type are similar to the equations found in Electrical Science, involving for instance three currents I_1, I_2 and I_3, that have been found by resolving the current flow at any point in a circuit.

The following examples illustrate possible steps but each set of equations can be tackled differently.

Main Example

Let's consider the following system:

$$\mathscr{S}:\begin{cases} x+ \quad 2y+ \quad z \quad =8 & \text{eq.1}\\ 2x+ \quad y- \quad z \quad =1 & \text{eq.2}\\ x- \quad y+ \quad 3z \quad =8 & \text{eq.3}\end{cases}$$

We can rearrange eq. 1 for z:

$$x+2y+x=8 \Leftrightarrow z=8-x-2y$$

And we can now substitute it in Eq. 2 and 3.

➔ Equation 2 becomes:

$$\begin{aligned} 2x+y-z=1 \quad &\Rightarrow \quad 2x+y-(8-x-2y)=1 && \text{substitution } z\leftarrow 8-x-2y\\ &\Rightarrow \quad 2x+y+x+2y-8=1 && \text{removing the brackets}\\ &\Rightarrow \quad 3x+3y=9 && \text{rearranging}\end{aligned}$$

Let's name $3x+3y=9$ equation 4.

➔ Equation 3 becomes:

$$\begin{aligned} x-y+3z=8 \quad &\Rightarrow \quad x-y+3(8-x-2y)=8 && \text{substitution } z\leftarrow 8-x-2y\\ &\Rightarrow \quad x-y-3x-6y+24=8 && \text{removing the brackets}\\ &\Rightarrow \quad -2x-7y=-16 && \text{rearranging}\end{aligned}$$

Let's name $-2x-7y=-16$ equation 5.

We now have 2 equations in 2 unknowns:

$$\mathscr{S}_{small}:\begin{cases} 3x+ \quad 3y \quad =9 & \text{eq.4}\\ -2x- \quad 7y \quad =-16 & \text{eq.5}\end{cases}$$

that we can solve !

Main Example

Let's now solve the small system: If we:

- ➔ multiply eq.4 by 2, we will have $6x$:

$$3x+3y=9 \Leftrightarrow 6x+6y=18$$

- ➔ multiply eq.5 by 3, we will have $-6x$:

$$-2x-7y=-16 \Leftrightarrow -6x-21y=-48$$

Adding the two gives:

- ➔ Right hand side:

$$\underbrace{6x+6y}_{\text{2 times eq. 4}} + \underbrace{-6x-21y}_{\text{2 times eq.5}} = -15y$$

- ➔ Left hand side:

$$\underbrace{18}_{\text{2 times eq. 1}} + \underbrace{-48}_{\text{3 times eq.2}} = -30$$

So $-15y=-30 \Leftrightarrow y=2$. Using eq. 4 to find x (we could have used eq. 5!):

$$\begin{aligned} 3x+3y=9 &\Rightarrow 3x+3(2)=9 && \text{substitution } y \leftarrow 2 \\ &\Rightarrow 3x=9-6 && \text{x as the subject} \\ &\Rightarrow x=1 \end{aligned}$$

We now know both x and y! To identify z, let's use eq. 1 (we could use eq. 2 or 3).

$$\begin{aligned} x+2y+z=8 &\Rightarrow (1)+2(2)+z=8 && \text{substitutions } x \leftarrow 1 \text{ and } y \leftarrow 2 \\ &\Rightarrow z=8-1-4 && \text{z as the subject} \\ &\Rightarrow z=3 \end{aligned}$$

As a check the values we have found should satisfy both equations:

- ➔ eq. 1: $x+2y+z=8 \Rightarrow 1\times1+2\times2+1\times3=8$ as required.
- ➔ eq. 2: $2x+y-z=1 \Rightarrow 2\times1+1\times2-1\times3=1$ as required.
- ➔ eq. 3: $x-y+3z=8 \Rightarrow 1\times1-1\times2+3\times3=8$ as required.

Let's consider the following system:

$$\mathscr{S}: \begin{cases} I_1+I_2=I_3 & \text{eq.1} \\ 4I_2=3I_3 & \text{eq.2} \\ 3I_1+5I_2=10.8 & \text{eq.3} \end{cases}$$

On inspection it can be seen that the three variables do not appear in all three equations. This is common in situations like this, and the first equation sums all three , again this is usual.

Going further

Note that we could have rewritten $\mathscr{S}$ as:

$$\mathscr{S}:\begin{cases} I_1+ & I_2- & I_3 & =0 & \text{eq.1}\\ 0I_1+ & 4I_2- & 3I_3 & =0 & \text{eq.2}\\ 3I_1+ & 5I_2+ & 0I_3 & =10.8 & \text{eq.3}\end{cases}$$

When a variable does not appear, it is equivalent as having it multiplied by 0.

We can use eq.2 to eliminate I_3 from eq.1: We start by rearranging eq. 2 with I_3 as the subject:

$$4I_2 = 3I_3 \Leftrightarrow I_3 = \frac{4}{3}I_2$$

We substitute I_3 with $\frac{4}{3}I_2$ in eq. 1:

$$\begin{aligned} I_1+I_2=I_3 \quad &\Rightarrow \quad I_1+I_2=\frac{4}{3}I_2 && \text{substitution } I_3 \leftarrow \tfrac{4}{3}I_2\\ &\Rightarrow \quad I_1=\frac{1}{3}I_2 && \text{rearranging the } I_2\text{s}\end{aligned}$$

Let's name $I_1 = \frac{1}{3}I_2$ equation 4. Eq. 3 and eq. 4 only contain I_1 and I_2. So we can solve these using methods introduced in Sec. II. In the present case, it worth substituting I_1 in eq.3, using eq.4 relationship $I_1 = \frac{1}{3}I_2$:

$$\begin{aligned} 3I_1+5I_2=10.8 \quad &\Rightarrow \quad 3(\frac{1}{3}I_2)+5I_2=10.8 && \text{substitution } I_1 \leftarrow \tfrac{1}{3}I_2\\ &\Rightarrow \quad 6I_2=10.8 && \text{rearranging the } I_2\text{s}\\ &\Rightarrow \quad I_2=1.8 && \text{dividing both side by 6}\end{aligned}$$

Using eq. 4 to find I_1:

$$\begin{aligned} I_1=\frac{1}{3}I_2 \quad &\Rightarrow \quad I_1=(\frac{1}{3}1.8) && \text{substitution } I_2 \leftarrow 1.8\\ &\Rightarrow \quad I_1=0.6 \end{aligned}$$

And now, we can use either eq. 1 or eq. 2 to find I_3 (we cannot use eq. 3, it does not contain I_3!). Using eq. 2:

$$\begin{aligned} 4I_2=3I_3 \quad &\Rightarrow \quad 4(1.8)=3I_3 && \text{substitution } I_2 \leftarrow 1.8\\ &\Rightarrow \quad I_3=2.4 && I_3 \text{ as the subject}\end{aligned}$$

We can check with eq. 1:
$I_1+I_2=I_3 \Rightarrow 0.6+1.8=2.4$ as required.

Exercise 8.

Solve the following simultaneous equations in 3 unknowns

8.1

$$\mathscr{S}_1:\begin{cases} I_2+I_3 &= I_1\\ 4I_2 &= 3I_3\\ 3I_1+5I_2 &= 2.88\end{cases}$$

8.2

$$\mathscr{S}_2:\begin{cases} I_2+I_3 &= I_1\\ I_2 &= 2I_3\\ 2I_1+3I_3 &= 2.7\end{cases}$$

8.3

$$\mathscr{S}_3:\begin{cases} I_2+I_3 &= I_1\\ 3I_2+I_3 &= 3.2\\ 5I_1+I_3 &= 7.5\end{cases}$$

8.4

$$\mathscr{S}_4:\begin{cases} 2x+ & 3y- & z & =9\\ x- & 2y+ & z & =-3\\ 3x- & y+ & 2z & =4\end{cases}$$

8.5

$$\mathscr{S}_5 : \begin{cases} x + & 2y + & z & = 6 \\ 2x - & 4y + & z & = -4 \\ -3x - & y + & 2z & = 9.5 \end{cases}$$

8.6

$$\mathscr{S}_6 : \begin{cases} x + & 3y + & z & = 5 \\ x - & 2y - & 2z & = 7 \\ 2x + & y - & 4z & = 24 \end{cases}$$

8.7

$$\mathscr{S}_7 : \begin{cases} x - & 3y + & 2z & = -19 \\ 2x - & 2y + & z & = -15 \\ -x + & y - & 3z & = 20 \end{cases}$$

8.8

$$\mathscr{S}_8 : \begin{cases} 4x + & 2y + & 3z & = 36 \\ 2x - & 3y + & 2z & = 17 \\ -5x - & y + & 2z & = -9 \end{cases}$$

8.9

$$\mathscr{S}_9 : \begin{cases} 5x + & 3y + & z & = 36 \\ 3x + & y - & 2z & = 33 \\ 2x + & 6y - & 3z & = 41 \end{cases}$$

VI Solutions to exercises

Solution 1.

1.1 $x = 4, y = 2$

1.2 $x = 3, y = -5$

1.3 $x = -3, y = -4$

1.4 $x = 2, y = 1$

1.5 $x = 1, y = \frac{1}{5}$

1.6 $x = 3, y = 2$

Solution 2.

2.1 $x = 7, y = 1$

2.2 $x = 3, y = 1$

2.3 $x = 6, y = 2$

2.4 $x = 2, y = 4$

2.5 $x = 4, y = -3$

2.6 $x = 3, y = 1.5$

2.7 $x = 2, y = 1$

2.8 $x = 1, y = \frac{1}{5}$

2.9 $x = 3, y = 2$

2.10 $x = 2, y = 2$

2.11 $x = 3, y = 6$

2.12 $x = 9.5, y = 2$

Solution 3.

3.1 $x = 2, y = 4$

3.2 $x = 4, y = 2$

3.3 $x = 6, y = 2$

Solution 4.

4.1 $x = 5, y = 2$

4.2 $x = 2, y = 3$

4.3 $x = \frac{4}{13}, y = -\frac{62}{13}$

4.4 $x = 2, y = \frac{3}{2}$

4.5 $x = 9, y = 4$

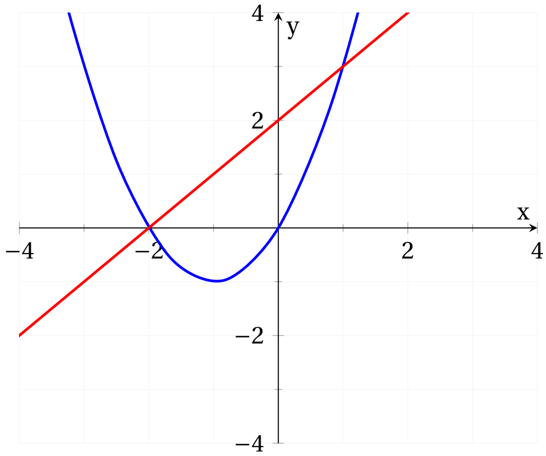
4.6 $x = 2, y = \frac{3}{2}$

Solution 5.

5.1 $x_1 = 1, y_1 = 2$ and $x_2 = -4, y_2 = 12$

5.2 $x_1 = 1, y_1 = 0$ and $x_2 = -6, y_2 = 42$

Solution 6.



$x_1 = -2, y_1 = 0$ and $x_2 = 1, y_2 = 3$

Solution 7.

7.1 $x_1 = 1, y_1 = -2$ and $x_2 = -\frac{1}{7}, y_2 = \frac{10}{7}$

7.2 $x_1 = 3, y_1 = 5$ and $x_2 = -\frac{5}{2}, y_2 = 6$

7.3 $x_1 = -1, y_1 = -1$ and $x_2 = \frac{19}{11}, y_2 = -\frac{1}{11}$

7.4 $x = 1, y = 2$

7.5 $x_1 = -2, y_1 = 6$ and $x_2 = 4, y_2 = -3$

7.6 $x_1 = 2, y_1 = 15$ and $x_2 = 5, y_2 = 6$

7.7 $x_1 = 3, y_1 = 5$ and $x_2 = -\frac{5}{4}, y_2 = -12$

7.8 $x_1 = 3, y_1 = \frac{2}{3}$ and $x_2 = -\frac{4}{3}, y_2 = -\frac{3}{2}$

Solution 8.

8.1 $I_1 = 0.56, I_2 = 0.24, I_3 = 0.32$

8.2 $I_1 = 0.9, I_2 = 0.6, I_3 = 0.3$

8.3 $I_1 = 1.4, I_2 = 0.9, I_3 = 0.5$

8.4 $x = 1, y = 3, z = 2$

8.5 $x = -1, y = 1.5, z = 4$

8.6 $x = 3, y = 2, z = -4$

8.7 $x = -3, y = 2, z = -5$

8.8 $x = 4, y = 1, z = 6$

8.9 $x = 7, y = 2, z = -5$

Bibliography

[Croft and Davidson, 2016] Croft, A. and Davidson, R. (2016). *Foundation Maths*. Pearson.