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Introduction to Matrices

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As always, please free to refer to the book [Croft and Davidson, 2016] for details.

I Introduction

A matrix organizes a group of numbers, or variables, with specific rules of arithmetic. They are used in many domains, and are central in engineering, computer sciences, mathematics, etc.

For instance, graphic softwares uses matrix mathematics to process linear transformations to render images, for instance, flipping an image. In a video game, this is used to render the upside-down mirror image of a landscape reflected in a lake.

Def.

MATRIX: A matrix is an array of numbers (or symbols) arranged in rows and columns, similarly to a table, for instance:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{pmatrix}$$

They are usually noted with a capital letter.

Here is a few examples of matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 1 & x \\ y & 2 \\ 2 & x \end{pmatrix}, C = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -4 & 9 \\ 2 & 1 & 0 \end{pmatrix}$$

The numbers or symbols are called elements. For instance, the matrix A:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

has four elements. There is several way of noting the elements of a matrix. For instance, for identifying the second element of the first column of A:

- ➔ $A(1,2) = 2$ used in computer science
- ➔ $a_{12} = 2$ used in maths
- ➔ $A_{12} = 2$ used in engineering

It is always first the rows, then the column. In these notes, we will use preferably the second notation.

A matrix is defined by the number of rows and columns it has, this gives us the size (or order) of the matrix. For instance, if f matrix A has two rows and three columns, it is of size 2×3. Similarly to identifying an element, for giving the dimensions, it is always first the number of rows, then the number of columns.

Def.

SQUARE AND RECTANGULAR MATRIX: A square matrix has the same number of rows and column.
If a matrix is not square, it is rectangular.
A 2×2 matrix is hence square, while a 2×3 matrix is rectangular.

II Equality of Matrices, addition and subtraction

II a) Equality

Matrices are equal if and only if:

1. they are of the same order

2. their corresponding elements are equal
   corresponding elements mean elements which are in the same position

Main Example

If

$$A = \begin{pmatrix} 1 & -3 \\ 4 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & -3 \\ 4 & 2 \end{pmatrix}$$

A is a 2×2 matrix, and B is a 2×2 matrix. We have $a_{11} = b_{11}, a_{12} = b_{12}, a_{21} = b_{21}$ and $a_{22} = b_{22}$. We can then say that A = B.

If

$$C = \begin{pmatrix} -1 & -1 \\ 2 & 6 \end{pmatrix}, D = \begin{pmatrix} -1 & -1 \\ -1 & 6 \end{pmatrix}$$

then $C \neq D$, as $c_{21} \neq d_{21}$.

If

$$E = \begin{pmatrix} 3 & y \\ -3 & 3 \end{pmatrix}, F = \begin{pmatrix} 3 & 1 \\ -3 & x \end{pmatrix}$$

and if we are told that E = F, then $x = 3$ and $y = 1$.

If

$$G = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}, H = \begin{pmatrix} 2 & 4 \\ -1 & 3 \\ -2 & -1 \end{pmatrix}$$

then $G \neq H$ as they do not have the same size.

II a) i Addition and subtraction

Addition (and subtraction) can only be carried out on matrices that are the same size. To add (or subtract) two matrices simply add (or subtract) the corresponding elements.

Main Example

If

$$A = \begin{pmatrix} 1 & -3 \\ 4 & 2 \end{pmatrix}, B = \begin{pmatrix} 4 & -2 \\ 5 & -1 \end{pmatrix}$$

and let's define C = A + B. Then, $C_{11} = A_{11} + B_{11}$, $C_{12} = A_{12} + B_{12}$, $C_{21} = A_{21} + B_{21}$ and $C_{22} = A_{22} + B_{22}$:

$$C = A + B = \begin{pmatrix} 1+4 & -3+(-2) \\ 4+5 & 2+(-1) \end{pmatrix} = \begin{pmatrix} 5 & -5 \\ 9 & 1 \end{pmatrix}$$

If

$$C = \begin{pmatrix} 1 & 5 & 8 \end{pmatrix}, D = \begin{pmatrix} 0 & -3 & 1 \end{pmatrix}$$

then

$$C - D = \begin{pmatrix} 1-0 & 5-(-3) & 8-1 \end{pmatrix} = \begin{pmatrix} 1 & 8 & 7 \end{pmatrix}$$

Exercise 1.
Given that:

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 7 \end{pmatrix}, B = \begin{pmatrix} 3 & 0 \\ 5 & -3 \end{pmatrix}, C = \begin{pmatrix} 1 & 3 & 4 \\ 3 & 0 & -3 \\ 5 & 2 & 7 \end{pmatrix}, D = \begin{pmatrix} 2 & -3 & 1 \\ -1 & 1 & 4 \\ 2 & 2 & 5 \end{pmatrix}$$

Calculate, if possible:

1.1 A + B	1.2 A − B	1.3 B − A
1.4 A − C	1.5 C + D	1.6 C − D
1.7 D − C		

III Multiplication

III a) Multiplication by a scalar

To multiply a matrix by a scalar value (ie a number), simply multiply each element in the matrix by the scalar quantity

Main Example

If

$$A = \begin{pmatrix} 1 & -3 \\ 4 & 2 \end{pmatrix}$$

and let's define $C = 3A$. Then, $c_{11} = 3a_{11}$, $c_{12} = 3a_{12}$,$c_{21} = 3a_{21}$ and $c_{22} = 3a_{22}$:

$$C = 2A = \begin{pmatrix} 3\times 1 & 3\times(-3) \\ 3\times 4 & 3\times 2 \end{pmatrix} = \begin{pmatrix} 3 & -9 \\ 12 & 6 \end{pmatrix}$$

If

$$B = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 4 \end{pmatrix}$$

and let's define $D = 2B$. Then,

$$D = 2B = \begin{pmatrix} 2\times 2 & 2\times 1 & 2\times(-1) \\ 2\times 1 & 2\times 2 & 2\times 4 \end{pmatrix} = \begin{pmatrix} 4 & 2 & -2 \\ 2 & 4 & 8 \end{pmatrix}$$

III a) i Multiplication by another matrix

This is a bit more complex.

Def.

MATRIX MULTIPLICATION: If $A = (a_{ij})$ is an $n \times p$ matrix and $B = (b_i j)$ is a $p \times q$, then the product $C = (c_{ij})$ of A and B, $C = A \times B$:

- ➔ C is a $n \times q$ matrix
- ➔ $c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$

See Fig. 1 for a visual scheme of this formula.
Note that matrix multiplication is a reduction! From two terms, you have only one after. C's dimensions are the first one of A and the last one of B. It also means that the last dimension of A and the first dimension of B have to be the same.

Okay, this formula seems horrible, right ? It is easier with a picture, see Fig. 1.

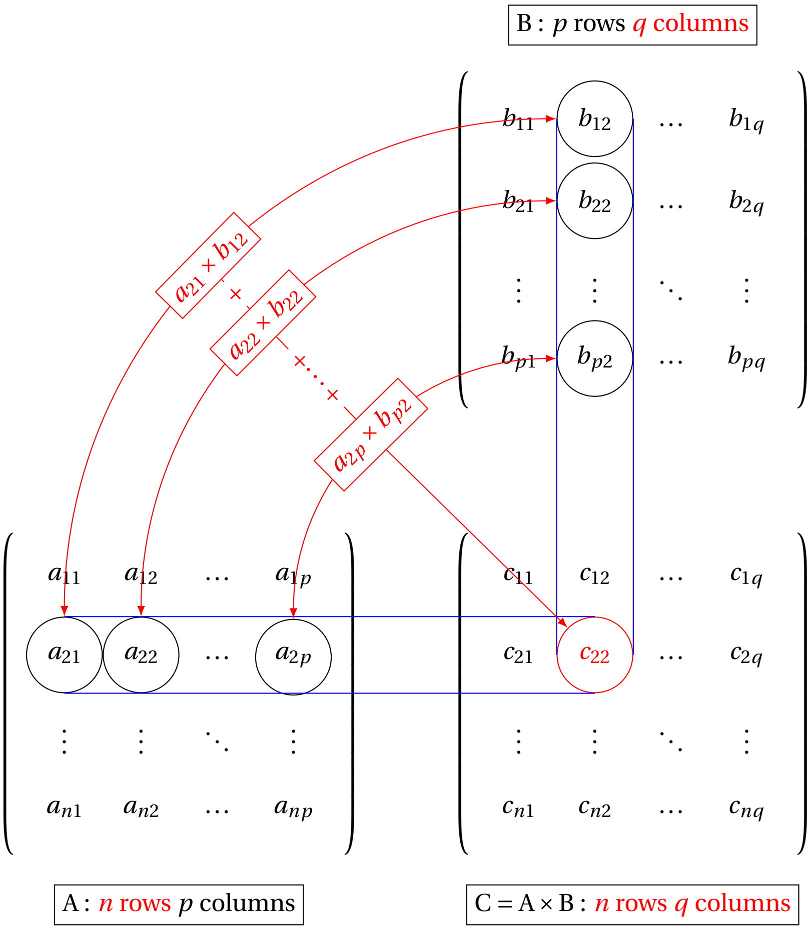

Figure 1: Sketch on how to multiply two matrices.

Main Example

Let's have:

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 5 & 8 \end{pmatrix}, B = \begin{pmatrix} 7 & 6 \\ 10 & 9 \end{pmatrix}$$

➔ Can we calculate A × B ?

A × B cannot be done!

As we go across the first row in A and down the first columns in B, we can see that there are not enough numbers in B:

$$\begin{aligned} \text{first element} = \; & a_{11} \times b_{11} + && \text{check, it is } 2 \times 7 \\ & a_{12} \times b_{21} + && \text{check, it is } 3 \times 10 \\ & ??? && \text{we have the } a_{13} = 4 \text{ but no } b_{31} \text{ !!} \end{aligned}$$

We have 3 numbers in the rows of A but only 2 in the columns B. A is a 2 × 3 matrix, while B is a 2 × 2 matrix.

A and B are not compatible.

➔ Can we calculate B × A?

The second dimension of B is 2, and the first dimension of A is also 2.

B and A are compatible.

$$B \times A = \begin{pmatrix} 7 \times 2 + 6 \times 1 & 7 \times 3 + 6 \times 5 & 7 \times 4 + 6 \times 8 \\ 10 \times 2 + 9 \times 1 & 10 \times 3 + 9 \times 5 & 10 \times 4 + 9 \times 8 \end{pmatrix} = \begin{pmatrix} 20 & 51 & 76 \\ 29 & 75 & 112 \end{pmatrix}$$

Tip

This example shows that A × B and B × A are most of the time not equal ! Matrix multiplication is said to be *not commutative*.
Be careful of the order when multiplying matrices.

III a) ii But... Why ?

Let's see why it is pretty cool, through a few examples.

III a) ii 1 A store Suppose a game store sells four types of product: Video Games (for a price of 50$ each), Comics (for $ each), Magic cards (for 4$ per pack) and candy bars (a bit of chocolate cost 1$). And let's suppose they are open five days per week: Monday to Friday. The quantity of stuff they sell per week is gathered in Tab. 1.

Table 1: Quantities sold by the store

	Video Games	Comics	Magic Cards	Candy Bars
Monday	5	33	55	201
Tuesday	4	42	20	192
Wednesday	8	55	25	212
Thursday	2	18	22	181
Friday	6	45	75	221

How much did they make on Monday ? The number of Video Games, Comics, Magic cards and candy bars multiplied by their respected values.

$$\text{sales Monday} = 5 \times 50\$ + 33 \times 12\$ + 55 \times 4\$ + 201 \times 1\$ = 1067\$$$

and, for the other days:

$$\text{sales Tuesday} = 4 \times 50\$ + 42 \times 12\$ + 20 \times 4\$ + 192 \times 1\$ = 976\$$$

$$\text{sales Wednesday} = 8 \times 50\$ + 55 \times 12\$ + 25 \times 4\$ + 212 \times 1\$ = 1372\$$$

$$\text{sales Thursday} = 2 \times 50\$ + 18 \times 12\$ + 22 \times 4\$ + 181 \times 1\$ = 585\$$$

$$\text{sales Friday} = 6 \times 50\$ + 45 \times 12\$ + 75 \times 4\$ + 221 \times 1\$ = 1361\$$$

These formulas are the same as the formula (the infamous $c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$) for multiplying matrices !

So, if the "quantity" matrix is Q, and the prices are in the matrix p:

$$Q = \begin{pmatrix} 5 & 33 & 55 & 201 \\ 4 & 42 & 20 & 192 \\ 8 & 55 & 25 & 212 \\ 2 & 18 & 22 & 181 \\ 6 & 45 & 75 & 221 \end{pmatrix}, \quad p = \begin{pmatrix} 50 \\ 12 \\ 4 \\ 1 \end{pmatrix}$$

Then the sales s are simply $s = Q \times p$.

$$s = Q \times p = \begin{pmatrix} 5 \times 50 + 33 \times 12 + 55 \times 4 + 201 \times 1 & = 106 \\ 4 \times 50 + 42 \times 12 + 20 \times 4 + 192 \times 1 & = 976 \\ 8 \times 50 + 55 \times 12 + 25 \times 4 + 212 \times 1 & = 1372 \\ 2 \times 50 + 18 \times 12 + 22 \times 4 + 181 \times 1 & = 585 \\ 6 \times 50 + 45 \times 12 + 75 \times 4 + 221 \times 1 & = 1361 \end{pmatrix}$$

The produce of matrices Q and p gives the sales ! This is one of the reason the product is define in such a nasty way.

> **Tip**
> Look at Tab. 1, and at the matrix Q.
> They are basically the same ! Matrices and tables are really close. A (good!) way to see matrices is to consider them as spreadsheets.

[[TODO check: https://betterexplained.com/articles/linear-algebra-guide/]]

[[TODO linear maps ?]]

Exercise 2.
Given

$$A = \begin{pmatrix} 1 & 2 \\ 5 & 7 \end{pmatrix}, B = \begin{pmatrix} 4 & -2 \\ 1 & 2 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix},$$

calculate:

2.1 $2A+B$	2.2 $A-3B$	2.3 $A+B+2C$
2.4 AB	2.5 BA	2.6 AC

Exercise 3.
Given

$$A = \begin{pmatrix} 3 & -2 & 5 \\ 2 & 1 & 4 \end{pmatrix}, B = \begin{pmatrix} 3 & 2 \\ 5 & 0 \\ -3 & -1 \end{pmatrix}, C = \begin{pmatrix} 1 & 2 & 4 \\ -2 & 0 & 1 \\ 1 & -1 & 5 \end{pmatrix}, D = \begin{pmatrix} 1 & 3 & 4 \end{pmatrix},$$

and

$$E = \begin{pmatrix} -3 & 1 \\ 2 & 5 \end{pmatrix}, F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, G = \begin{pmatrix} 1 & -2 & -1 \\ 2 & -5 & 3 \\ 4 & -3 & 5 \end{pmatrix},$$

calculate the following, if they cannot be calculated state why:

3.1 AB	3.2 AC	3.3 BA	3.4 AE
3.5 DB	3.6 BD	3.7 CF	3.8 FC
3.9 DF	3.10 EA	3.11 CG	3.12 GC
3.13 $C^2 = C \times C$	3.14 $C^3 = C \times C \times C$		

IV Identity matrix, Division of Matrices and Inverse

IV a) Identity matrix

The identity matrix acts the same way in matrix multiplication as the number 1 does in multiplication of numbers. For instance, $1 \times 5 = 5$, and $3 \times 1 = 3$. We want to identify a matrix I so $M \times I = M$ and $I \times M = M$, for *all* the matrices M. This matrix, the identity matrix, has no "effect" in matrix multiplication in the same way the number 1 has no effect in multiplication. If

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, I = \begin{pmatrix} i_{11} & i_{12} \\ i_{21} & i_{22} \end{pmatrix},$$

and $M \times I = M$, it means that:

$$\begin{cases} m_{11} \times i_{11} + m_{12} \times i_{21} = m_{11} \\ m_{11} \times i_{12} + m_{12} \times i_{22} = m_{12} \\ m_{21} \times i_{11} + m_{22} \times i_{21} = m_{21} \\ m_{21} \times i_{12} + m_{22} \times i_{22} = m_{11} \end{cases}$$

The solution is $i_{11} = 1, i_{12} = 0, i_{21} = 0, i_{22} = 1$, which means that

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

Def.

THE IDENTITY MATRIX: The identity matrix is usally noted I.

- ➔ It is always a square matrix (2×2, 3×3 etc.)
- ➔ It has one's on the leading diagonal and zeros everywhere else!

When sizes are compatible, for any matrix M:

- ➔ $M \times I = M$
- ➔ $I \times M = M$

IV b) Division of Matrices

Division of matrices is not possible, only addition, subtraction and multiplication.

But it is not the end: if division was not possible in the number system, one could instead multiplying by what is known as the inverse.

Tip

In the matrix world, it is not possible to divide by a matrix A.
Instead, we multiply by the inverse of A.

For instance, it is possible to multiply by $\frac{1}{5}$ instead of dividing by 5. $\frac{1}{5}$ is called the multiplicative inverse of 5. Note that $\frac{1}{5} \times 5 = 1$. This inverse is noted A^{-1}.

Def.

INVERSE MATRIX: A matrix B is the inverse of A if and only if:

- ➔ $A \times B = I$
- ➔ $B \times A = I$

B is then the inverse of A and can be noted A^{-1}.

So let's see if we can find a matrix inverse of A.

IV b) i Inverse of a 2×2 matrix

So let's have the matrix A:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and suppose that A^{-1} is:

$$A^{-1} = \begin{pmatrix} e & f \\ g & h \end{pmatrix},$$

Because we have $A \times A^{-1} = I$, it means that:

$$\begin{cases} a \times e + b \times g = 1 \\ a \times f + b \times h = 0 \\ c \times e + d \times g = 0 \\ c \times f + d \times h = 1 \end{cases}$$

It does not seems easy to solve. Let's use a trick, and let's consider the matrix B:

$$B = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

And let's calculate $C = AB$. Its first element is:

$$\begin{aligned} c_{11} &= a_{11} \times b_{11} + a_{12} \times b_{21} \\ &= a \times \left(\frac{1}{ad - bc} d \right) + b \times \left(\frac{1}{ad - bc} - c \right) \\ &= \frac{ad}{ad - bc} + \frac{-bc}{ad - bc} \\ &= 1 \end{aligned}$$

Its second element is:

$$\begin{aligned} c_{12} &= a_{11} \times b_{12} + a_{12} \times b_{22} \\ &= a \times \left(\frac{1}{ad - bc} - c \right) + c \times \left(\frac{1}{ad - bc} a \right) \\ &= \frac{-ac}{ad - bc} + \frac{ac}{ad - bc} \\ &= 0 \end{aligned}$$

Its third element is:

$$\begin{aligned} c_{21} &= a_{21} \times b_{11} + a_{22} \times b_{21} \\ &= c \times \left(\frac{1}{ad - bc} d \right) + d \times \left(\frac{1}{ad - bc} - c \right) \\ &= \frac{-cd}{ad - bc} + \frac{cd}{ad - bc} \\ &= 0 \end{aligned}$$

And finally the last element is:

$$\begin{aligned} c_{22} &= a_{21} \times b_{12} + a_{22} \times b_{22} \\ &= c \times \left(\frac{1}{ad - bc} - b \right) + d \times \left(\frac{1}{ad - bc} a \right) \\ &= \frac{-bc}{ad - bc} + \frac{ad}{ad - bc} \\ &= 1 \end{aligned}$$

It means that

$$C = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

It is easy to check that $B \times A = I$, it means that $B = A^{-1}$ and is the inverse of A.

Because we divide by $ad - bc$, it means that this number has to be different from 0. This is why it has a special name.

Def.

DETERMINANT: The Determinant of the 2×2 matrix A:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

is the number $ad - bc$. If it is not 0, then the matrix has an inverse ! It is can be noted in several way:

- ➔ $\det A$
- ➔ $|A|$
- ➔ $\det(A)$

Tip

If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then the inverse of A is found by

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

Or:

- ➔ Write A
- ➔ exchange the a and the d
- ➔ change the sign of the b and the c
- ➔ multiply by one over the determinant $\det A$

For instance, if

$$A = \begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 6 & 3 \end{pmatrix},$$

then

$$\det A = 2 \times 4 - 1 \times 5 = 8 - 5 = 3$$

$\det A \neq 0$ it means that A has an inverse ! and, for B:

$$\det B = 1 \times 3 - 6 \times 2 = 3 - 12 = -9$$

$\det B \neq 0$ it means that B also has an inverse !

Now we can calculate A^{-1} and B^{-1}:

$$A^{-1} = \frac{1}{3} \begin{pmatrix} 4 & -5 \\ -1 & 2 \end{pmatrix}, B^{-1} = -\frac{1}{9} \begin{pmatrix} 3 & -2 \\ -6 & 1 \end{pmatrix},$$

Now if you multiply a matrix by its inverse you should get the identity matrix. Do this to check with A, B and their inverses if it is correct.

Exercise 4. Determinant of a product of matrices
Let's prove that $\det(M \times N) = \det(M)\det(N)$, for any matrices M and N ! If we define

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, N = \begin{pmatrix} c & d \\ e & f \end{pmatrix},$$

4.1 Verify that $M \times N = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$.

4.2 Verify that $\det(M \times N) = adeh + bcfg - bceh - adfg$

4.3 Conclude that $\det(M \times N) = \det(M)\det(N)$.

Exercise 5.

5.1 Find the determinants and inverses of:

$$A = \begin{pmatrix} 4 & 3 \\ 1 & -2 \end{pmatrix}, B = \begin{pmatrix} 5 & 2 \\ -3 & 4 \end{pmatrix}, C = \begin{pmatrix} 2 & 4 \\ -1 & 2 \end{pmatrix},$$

and

$$D = \begin{pmatrix} 1 & 1 \\ 3 & 6 \end{pmatrix}, E = \begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}, F = \begin{pmatrix} -1 & 4 \\ 2 & -1 \end{pmatrix}$$

5.2 Let's have

$$G = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$$

Is it possible to find the inverse of the matrix G?

5.3 Let's have

$$H = \begin{pmatrix} 1 & 2 \\ -1 & 4 \\ 2 & 5 \end{pmatrix}$$

Is it possible to find the inverse of the matrix H?

V Solution of Simultaneous Equations using Matrices

V a) Inverse Matrix method

To solve equations of the form

$$\begin{cases} ax + by = p \\ cx + dy = q \end{cases}$$

for x and y: Re-write the equations in matrix form, by putting the coefficients of x and y (ie the numbers in front of x and y) into a matrix, thus ;

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

Multiplying these matrices out should bring you back to the original equations!

The equations are now in the general form:

$$A \times X = B$$

with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix}, B = \begin{pmatrix} p \\ q \end{pmatrix}$$

It would be convenient now to divide both sides by the matrix A !

But division is not possible for possible, so we multiply instead *both* sides by the inverse of A:

$$A^{-1}AX = A^{-1}B$$

The effect of this is that we can cancel the matrix A (and its inverse) on the left hand side. Indeed, multiplying a matrix by its inverse results in the identity matrix: It has no effect on the matrix X, as $I \times X = X$. As a result, we have:

$$X = A^{-1}B$$

We just have to calculate $A^{-1}B$ to have the solution, by reading x and y in $X = \begin{pmatrix} x \\ y \end{pmatrix}$.

Tip

This work for systems of large dimensions !
Even if for thousands of unknowns, you can solve a system using this method. Nevertheless, we will stick to equations with two unknowns.

Main Example

Let's solve:

$$\begin{cases} 3x & -2y & = & 6 \\ 2x & +y & = & 11 \end{cases}$$

for x and y.

1. Re-write the equations in matrix form:

$$\begin{pmatrix} 3 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \end{pmatrix}$$

2. What is A^{-1}?

   Division is not possible for possible, so we multiply instead *both* sides by the inverse of A:

$$A^{-1}AX = A^{-1}B$$

   The determinant of A is:

$$\det A = 3 - (-2 \times 2) = 7$$

   A is invertible and A^{-1} is:

$$A^{-1} = \frac{1}{7}\begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix}$$

3. We just have to calculate $A^{-1}B$ to have the solution:

$$A^{-1}B = \frac{1}{7}\begin{pmatrix} 1 \times 6 + 2 \times 11 = 28 \\ -2 \times 6 + 3 \times 11 = 21 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

4. by reading x and y in $X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, we have the solution: $x = 4$ and $y = 3$.

Let's verify now:

➔ $3x - 2y = 3 \times 4 - 2 \times 3 = 6$

➔ $2x + y = 2 \times 4 + 3 = 11$

et voilà !

Let's also solve:

$$\begin{cases} 4x & +3y & = & 5 \\ 2x & -y & = & 5 \end{cases}$$

for x and y:

1. Re-write the equations in matrix form:

$$\begin{pmatrix} 4 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

2. Identify A^{-1}

   The determinant of A is:

$$\det A = -1 \times 4 - (2 \times 3) = -10$$

   A is invertible and A^{-1} is:

$$A^{-1} = \frac{1}{7}\begin{pmatrix} -1 & -3 \\ -2 & 4 \end{pmatrix}$$

3. Calculate $A^{-1}B$ to have the solution:

$$A^{-1}B = \frac{1}{-10}\begin{pmatrix} -1\times 5 + -3\times 5 = -20 \\ -2\times 5 + 4\times 5 = 10 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

4. by reading x and y in $X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, we have the solution: $x = 2$ and $y = -1$.

V b) Cramers Rule

Let's see another way of solving simultaneous equations. To solve equations of the form

$$\begin{cases} ax & +by & = & p \\ cx & +dy & = & q \end{cases}$$

for x and y, we will follow the first steps as in Sec. V a) Re-write the equations in matrix form, by putting the coefficients of x and y (ie the numbers in front of x and y) into a matrix, thus ;

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

Multiplying these matrices out should bring you back to the original equations!

The equations are now in the general form:

$$A \times X = B$$

with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix}, B = \begin{pmatrix} p \\ q \end{pmatrix}$$

Prop.

CRAMERS RULES: Cramers rules allow to identify the solution of simultaneous equations:

$$AX = B$$

X is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.
Let's note A_i the matrix A where the ith column is replaced with B. For instance, if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B = \begin{pmatrix} p \\ q \end{pmatrix}$$

Then

$$A_1 = \begin{pmatrix} p & b \\ q & d \end{pmatrix}, A_2 = \begin{pmatrix} a & p \\ c & q \end{pmatrix}$$

Solving the system then involves calculating determinants:

- ➔ the determinant of the matrix A
- ➔ the determinant of each of the matrices A_i

Then the solution of the system is, for each x_i:

$$x_i = \frac{\det A_i}{\det A}$$

Here is a quick trick to understand Cramer's rules. It requires to know that the determinant of the product of two matrices M × N is actually the product of their determinant: $\det(M \times N) = \det(M) \times \det(N)$ (see Exercise. 4).
Let's consider $X_1 = \begin{pmatrix} x_1 & 0 \\ x_2 & 1 \end{pmatrix}$.
Then, we have

$$AX_1 = \begin{pmatrix} ax_1 + bx_2 & b \\ cx_1 + dx_2 & d \end{pmatrix} = \begin{pmatrix} p & b \\ q & d \end{pmatrix}$$

This last matrix is A_1 ! Now, we hence have $AX_1 = A_1$. We also know that $\det(AX_1) = \det(A)\det(X_1)$. Or, $\det(X_1) = x_1$.
It means that $\det(A)x_1 = \det(A_1)$! And hence

$$\frac{\det(A_1)}{\det(A)} = x_1$$

The other expression follows by using $X_2 = \begin{pmatrix} 1 & x_1 \\ 0 & x_2 \end{pmatrix}$.
Et voilà !

Exercise 6.
Solve the following sets of linear simultaneous equations using matrices firstly by the inverse matrix method and then by Cramer's rules.

6.1

$$\begin{cases} x - 3y = 1 \\ 2x + y = 9 \end{cases}$$

6.2

$$\begin{cases} 3x + 2y = 7 \\ 2x - 4y = 10 \end{cases}$$

6.3

$$\begin{cases} 4x + 3y = 13 \\ x - 2y = -5 \end{cases}$$

6.4

$$\begin{cases} 5x & +2y & = & 24 \\ -3x & +4y & = & -4 \end{cases}$$

6.5

$$\begin{cases} 3x & +y & = & -4 \\ 2x & -3y & = & 7 \end{cases}$$

6.6

$$\begin{cases} 2x & -y & = & 13 \\ 4x & +3y & = & 1 \end{cases}$$

6.7

$$\begin{cases} 7x & +y & = & -9 \\ x & +2y & = & 8 \end{cases}$$

6.8

$$\begin{cases} 3x & -5y & = & 19 \\ 2x & +4y & = & -24 \end{cases}$$

6.9

$$\begin{cases} 2x & +y & = & 5 \\ 4x & +2y & = & 10 \end{cases}$$

VI Higher determinants

VII 3×3 matrices

Determinents exist for bigger matrices as well. Let's see the definition for calculating the determinant of a 3×3 matrix.

Def.

3×3 DETERMINANT: The rule to calculate the determinant in a 3×3 matrix is

1. add the product of each right diagonals (from top left to bottom right):

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh$$

2. subtract the product of each left diagonals (from top right to bottom left):

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = -ceg - bdi - afh$$

3. the determinant follows:

$$\det A = aei + bfg + cdh - ceg - bdi - afh$$

To complete the diagonal, pick the remaining terms.

Main Example

For instance, if

$$A = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 3 & 4 \\ 3 & -1 & 2 \end{pmatrix}$$

then the determinant is:

$$\det A = \begin{array}{ll} +1\times3\times2 & \text{first right diagonal} \\ +4\times4\times3 & \text{second right diagonal} \\ +3\times2\times-1 & \text{third right diagonal} \\ -3\times3\times3 & \text{first left diagonal} \\ -4\times2\times2 & \text{second left diagonal} \\ -1\times4\times-1 & \text{third left diagonal} \end{array} = 6+48+(-6)-27-16+4$$

and $\det A = 9$.

VII a) Expanding the determinant

Prop.

LAPLACE EXPANSION OF THE DETERMINANT: Another way to computer the determinant of a matrix is to expand it across the top row.
Let's name A_i the submatrix A where the first row and the ith column has been removed.
The determinant $\det A$ is the sum of sub-determinants $\det A_i$, weighted by $a_{1i} * (-1)^{i+1}$. If A is a $n \times n$ matrix:

$$\det A = \sum_{i=1}^{n} (-1)^{i+1} a_{1i} \det A_i$$

It is a bit barbaric and is best shown by an example.

$$\det A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = +1\times a\times \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} -1\times b\times \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} +1\times c\times \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

It corresponds to:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = +1\times a\times \begin{vmatrix} e & f \\ h & i \end{vmatrix} -1\times b\times \begin{vmatrix} d & f \\ g & i \end{vmatrix} +1\times c\times \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Et voilà !

Please note:

- ➔ the sign before each small determinant. It comes from the $(-1)^{i+1}$
- ➔ the weight coefficient. It is the one at the intersection of the two removed lines

Main Example

For instance, let's have

$$A = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 3 & 4 \\ 3 & -1 & 2 \end{pmatrix}$$

then the determinant of A (remember that $\det A$ can also be noted $|A|$) is:

$$\det A = +1 \times 1 \times \begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix} - 1 \times 4 \times \begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} + 1 \times 3 \times \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix}$$

Which is:

$$\begin{aligned} \det A &= 1(3 \times 2 - (-1) \times 4) - 4(2 \times 2 - 3 \times 4) + 3(2 \times (-1) - 3 \times 3) \\ &= 1(6+4) - 4(-8) + 3(-11) \\ &= 10 + 32 - 33 \end{aligned}$$

and one again, $\det A = 9$.

Exercise 7.
Find the determinants of the following matrices, using both the Laplace expansion and the formula:

7.1 $B = \begin{pmatrix} 2 & -4 & 1 \\ 1 & 2 & -1 \\ -3 & 5 & 6 \end{pmatrix}$

7.2 $C = \begin{pmatrix} 1 & 4 & 6 \\ 3 & -2 & 2 \\ 1 & 4 & 7 \end{pmatrix}$

7.3 $D = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 2 \\ 4 & -3 & 3 \end{pmatrix}$

7.4 $E = \begin{pmatrix} 1 & 3 & -1 \\ 4 & 2 & -3 \\ 2 & 4 & 0 \end{pmatrix}$

7.5 $F = \begin{pmatrix} 2 & 3 & 1 \\ 3 & -1 & 3 \\ 2 & 2 & 5 \end{pmatrix}$

7.6 $G = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 4 \\ 1 & 3 & 2 \end{pmatrix}$

VIII Solutions to exercises

Solution 1.

1.1 $A + B = \begin{pmatrix} 4 & 4 \\ 7 & 4 \end{pmatrix}$

1.2 $A - B = \begin{pmatrix} -2 & 4 \\ -3 & 10 \end{pmatrix}$

1.3 $B - A = \begin{pmatrix} 2 & -4 \\ 3 & -10 \end{pmatrix}$

1.4 $A - C$ does not exist as A and C do not have the same sizes.

1.5 $C + D = \begin{pmatrix} 3 & 0 & 5 \\ 2 & 1 & 1 \\ 7 & 4 & 12 \end{pmatrix}$

1.6 $C - D = \begin{pmatrix} -1 & 6 & 3 \\ 4 & -1 & -7 \\ 3 & 0 & 2 \end{pmatrix}$

1.7 $D - C = \begin{pmatrix} 1 & -6 & -3 \\ -4 & 1 & 7 \\ -3 & 0 & -2 \end{pmatrix}$

Solution 2.

2.1 $2A + B = \begin{pmatrix} 6 & 2 \\ 11 & 16 \end{pmatrix}$

2.2 $A - 3B = \begin{pmatrix} -11 & 7 \\ 2 & 1 \end{pmatrix}$

2.3 $A + B + 2C = \begin{pmatrix} 7 & 0 \\ 2 & 3 \end{pmatrix}$

2.4 $AB = \begin{pmatrix} 6 & 2 \\ 27 & 4 \end{pmatrix}$

2.5 $BA = \begin{pmatrix} -6 & -6 \\ 11 & 16 \end{pmatrix}$

2.6 $AC = \begin{pmatrix} -3 & 6 \\ -9 & 21 \end{pmatrix}$

Solution 3.

3.1 $AB = \begin{pmatrix} -16 & 1 \\ -1 & 0 \end{pmatrix}$

3.2 $AC = \begin{pmatrix} 12 & 1 & 35 \\ 4 & 0 & 29 \end{pmatrix}$

3.3 $BA = \begin{pmatrix} 13 & -4 & 23 \\ 15 & -10 & 25 \\ -11 & 5 & -19 \end{pmatrix}$

3.4 AE does not exist as their size are not compatible.

3.5 $DB = \begin{pmatrix} 6 & -2 \end{pmatrix}$

3.6 BD does not exist as their size are not compatible.

3.7 $CF = \begin{pmatrix} 1 & 2 & 4 \\ -2 & 0 & 1 \\ 1 & -1 & 5 \end{pmatrix}$

3.8 $FC = \begin{pmatrix} 1 & 2 & 4 \\ -2 & 0 & 1 \\ 1 & -1 & 5 \end{pmatrix}$

3.9 $DF = \begin{pmatrix} 1 & 3 & 4 \end{pmatrix}$

3.10 $EA = \begin{pmatrix} -7 & 7 & -11 \\ 16 & 1 & 30 \end{pmatrix}$

3.11 $CG = \begin{pmatrix} 21 & -24 & 25 \\ 2 & 1 & 7 \\ 19 & -12 & 21 \end{pmatrix}$

3.12 $GC = \begin{pmatrix} 4 & 3 & -3 \\ 15 & 1 & 18 \\ 15 & 3 & 38 \end{pmatrix}$

3.13 $C^2 = \begin{pmatrix} 1 & -2 & 26 \\ -1 & -5 & -3 \\ 8 & -3 & 28 \end{pmatrix}$

3.14 $C^3 = \begin{pmatrix} 31 & -24 & 132 \\ 6 & 1 & -24 \\ 42 & -12 & 3169 \end{pmatrix}$

Solution 4.

4.1 Calculate $M \times N$.

4.2 $\det(M \times N) = \cancel{acef} + adeh + bcfg + \cancel{bdgh} - \cancel{acef} - bceh - adfg - \cancel{bdgh}$

4.3 $\det(M)\det(N)$ is also equal to $adeh + bcfg - bceh - adfg$

Solution 5.

5.1

$$A^{-1} = \frac{1}{-11}\begin{pmatrix} -2 & -3 \\ -1 & 4 \end{pmatrix}, B^{-1} = \frac{1}{26}\begin{pmatrix} 4 & -2 \\ 3 & 5 \end{pmatrix}, C^{-1} = \frac{1}{8}\begin{pmatrix} 2 & -4 \\ 1 & 2 \end{pmatrix},$$

and

$$D^{-1} = \frac{1}{3}\begin{pmatrix} 6 & -1 \\ -3 & 1 \end{pmatrix}, E^{-1} = \frac{1}{12}\begin{pmatrix} 4 & -1 \\ 0 & 3 \end{pmatrix}, F^{-1} = \frac{1}{-7}\begin{pmatrix} -1 & -4 \\ -2 & -1 \end{pmatrix}$$

5.2 $\det G = 0$ so G is not invertible and has no inverse.

5.3 H is not a square matrix, so it is not invertible and has no inverse.

Solution 6.

6.1 $x = 4, y = 1$

6.2 $x = 1, y = -2$

6.3 $x = 1, y = 3$

6.4 $x = 4, y = 2$

6.5 $x = \frac{-5}{11}, y = \frac{29}{11}$

6.6 $x = 4, y = -5$

6.7 $x = -2, y = 5$

6.8 $x = -2, y = -5$

6.9 $x = 2, y = 1$

Solution 7.

7.1 $\det B = 57$ 7.2 $\det C = -14$ 7.3 $\det D = 19$ 7.4 $\det E = -18$

7.5 $\det F = -41$ 7.6 $\det G = 0$

Bibliography

[Croft and Davidson, 2016] Croft, A. and Davidson, R. (2016). *Foundation Maths.* Pearson.