TP Animation 3D 6

Introduction to dynamical system : Parametric pendulum

6.1 Objectives

We will focus on physical model.vThen we will hard code a simple physical (real) system. Impact of the algorithm on precision will be highlighted.

6.2 What we will study

Botafumeiro is a huge thurible. It is aroud a meter and half long. It is pretty heavy, as its weight, once loaded with embers and incense, is about 80 kg. Botafumeiro is suspendedto the dome of the Santiago de Compostela Cathedral, 30 meters from the floor. The main reason of such a bulky thurible was, during middle age, pilgrims. They were exhausted, sweaty, unwashed and then smelly. The use of air purifying artifact was not, indeed, for sacred purpose ! Height guys, the tiraboleiros, pull and push the rope to swing the thurible. At the

Figure 6.1 – Botafumeiro and tiraboleiros

top of the trajectory, the censer reachs no less than 21 meters ! And yet, tiraboleiros start with an initial slope of less than 10 degree, and they just pull the rope while the thurible is near the floor, and give rope while it is at his climax. How did thy amplify so much the movement ?

6.3 The system

Finaly, the otafumeiro is a huge pendulum, with a variable length. It can be simply modelize, as only the gravity, the action of air (viscosity) and the rope tension is acting on it :

$$m\vec{a} = \vec{T} + m\vec{g} + \vec{f} \tag{6.1}$$

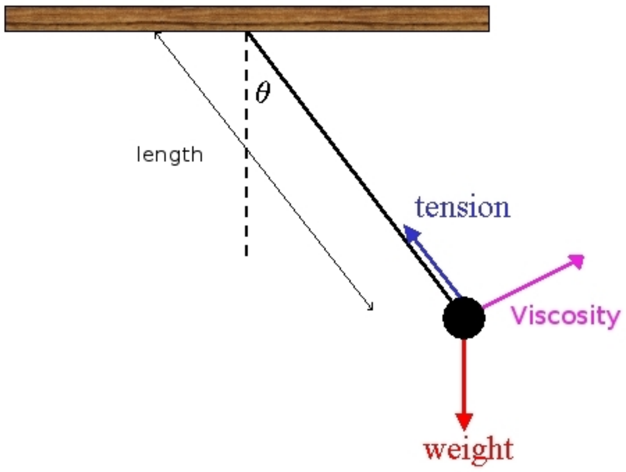


FIGURE 6.2 – Forces and notation

Each term may be rewritted.

$$\begin{aligned}
\vec{a} &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\vec{r}\right) \\
&= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(r\vec{\mathrm{e}}_r\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(r\dot{\theta}\vec{\mathrm{e}}_\theta + \dot{r}\vec{\mathrm{e}}_r\right) \\
&= \dot{r}\dot{\theta}\vec{\mathrm{e}}_\theta + r\ddot{\theta}\vec{\mathrm{e}}_\theta - r\dot{\theta}^2\vec{\mathrm{e}}_r + \ddot{r}\vec{\mathrm{e}}_r + \dot{r}\dot{\theta}\vec{\mathrm{e}}_\theta \\
&= \left(\ddot{r} - r\dot{\theta}^2\right)\vec{\mathrm{e}}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\vec{\mathrm{e}}_\theta
\end{aligned}$$

$$\vec{T} = T\vec{\mathrm{e}}_r$$

$$\vec{g} = g\cos\left(\theta\right)\vec{\mathrm{e}}_r + gsin\left(\theta\right)\vec{\mathrm{e}}_\theta$$

$$\begin{aligned}
\vec{f} &= -\lambda\vec{v} \\
&= -\lambda\frac{\mathrm{d}}{\mathrm{d}t}\left(r\vec{\mathrm{e}}_r\right) \\
&= -\lambda\left(r\dot{\theta}\vec{\mathrm{e}}_\theta + \dot{r}\vec{\mathrm{e}}_r\right)
\end{aligned}$$

Then we can decompose the expression (6.1) on each component. On $\vec{\mathrm{e}}_r$ we have :

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = T + m\cos\left(\theta\right) - \lambda\dot{r}$$

which can be reorganized as the expression of the tension T :

$$T = -m\cos\left(\theta\right) + \lambda\dot{r} + m\left(\ddot{r} - r\dot{\theta}^2\right).$$

On $\vec{e}_\theta$ we have :

$$m\left(2\dot{r}\dot{\theta}+r\ddot{\theta}\right)=mg\sin\left(\theta\right)-\lambda\dot{\theta}$$

which can be reorganized as a differential system by noting $\theta = y_1$ and $\dot{\theta} = y_2$:

$$\boxed{S=\left\{\begin{array}{lcl}\dot{y_1} & = & y_2\\ \dot{y_2} & = & \left(2\frac{\dot{r}}{r}-\frac{1}{rm}\lambda\right)y_2+\frac{g}{r}\sin y_1\end{array}\right.}$$

By compressing the notation, we can write :

$$\dot{y}=f(y)$$

6.4 How to solve the system ?

Let us have the following system :

$$\left\{\begin{array}{lcl}\dot{y} & = & f\left(y,t\right)\\ y\left(t_0\right) & = & y_0\end{array}\right.$$

In our case (with the pendulum expression previously seen), the time t will not appear. It remains in the bottom explanations, for the sake of generality.
There is two classical method :

Euler method The derivative is linearly approximated :

$$\dot{y}\left(t_{n+1}\right)\approx\frac{y\left(t_{n+1}\right)-y\left(t_n\right)}{\Delta t}$$

therefore we have :

$$\begin{array}{lcl}y\left(t_{n+1}\right) & \approx & \Delta t\dot{y}\left(t_{n+1}\right)+y\left(t_n\right)\\ & \approx & \Delta tf\left(y\left(t_n\right),t_{n+1}\right)+y\left(t_n\right)\\ & \approx & \Delta tf\left(y\left(t_n\right),t_n\right)+y\left(t_n\right)\end{array}$$

Therefore, the implemented expression will be :

$$\boxed{y\left(t_{n+1}\right)\approx\Delta tf\left(y\left(t_n\right),t_n\right)+y\left(t_n\right)}$$

Runge Kutta 4 method Runge Kutta method tries to estimate $\Delta y = y\left(t_{n+1}\right) - y\left(t_n\right)$, through a cascade of Euler methods.
It will be done with a weighted mean.

$$\Delta y\approx\frac{1}{6}\left(k_1+2k_2+2k_3+k_4\right)$$

The first increment is computed using the Euler method as time t_n. Therefore, the implemented expression will be :

$$\boxed{y\left(t_{n+1}\right)\approx y\left(t_n\right)+\frac{1}{6}\left(k_1+2k_2+2k-3+k4\right)}$$

$$k1=\Delta tf\left(y\left(t_n\right),t_n\right)$$

Next increment will be computed, with Euler method, as if time where $t_{n+\frac{1}{2}}$:

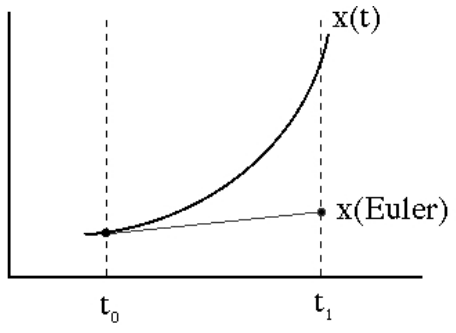
$$k2=\Delta tf\left(\underbrace{y\left(t_n\right)+\frac{1}{2}\Delta tk1}_{\text{half a step forward}},t_n+\frac{1}{2}\Delta t\right)$$

Then, this approximation is reinjected at time t_n to estimate again the derivative in time $t_{n+\frac{1}{2}}$:

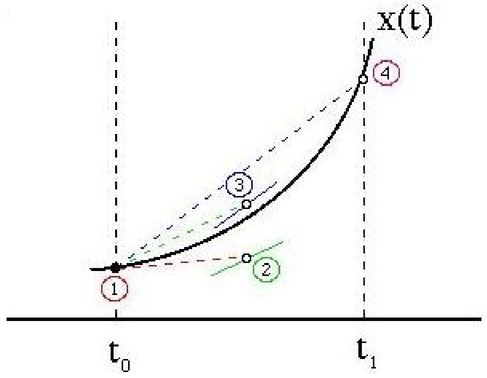
$$k3 = \Delta t f\left(\underbrace{y\left(t_n\right)+\frac{1}{2}\Delta t k2}_{\text{half a step forward}}, t_n+\frac{1}{2}\Delta t\right)$$

Last but not least, the derivative in t_{n+1} is estimated, using the last computed slope :

$$k4 = \Delta t f\left(\underbrace{y\left(t_n\right)+\Delta t k3}_{\text{a step forward}}, t_n+\Delta t\right)$$



(a) Euler



(b) RK 4

FIGURE 6.3 – Euler (a) and RK4 (b) method

6.5 Let's work !

One can note that only boxed equations are really important for you. The other equations are just for comprehension and for curious students.

6.5.1 How to get back to cartesian coordinates

Find how to go, from $\vec{e}_r$ and $\vec{e}_\theta$ polar coordinates, to the cartesian coordinates.

6.5.2 Implementation of integrators

Complete the code to compute and print the coordinates, with a deterministic law for r (of your choice. Sinus or constant laws are worth considering). You can choose the starting coordinates.

6.5.3 Display

Complete the code to display the pendulum, with a deterministic law for r.

6.5.4 Control the pendulum !

Complete the code to display the pendulum, with a mouse control for the length r !
Try to find how amplifying the amplitude of the botafumeiro :)