Coherent Structures, Particular Points and Models.

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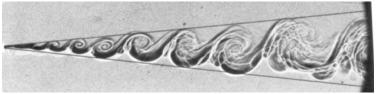
Hydrodynamic systems

Both open and confined flows are complex, and has potentially an infinite number of DoF, but coherent structures seem to play a major role.



Brown & Roshko, (1974), *J. Fluid Mech.*

What is a coherent structure (see *e.g.* Chassaing, Hussain, Lumley ...) ?

- spatially localized
- significant contribution to the kinetic energy
- significant life-time
- recurrent phenomenon
- material frontiers
- etc.

Von 'Heartman' street
Isla Socorro ($Re > 10^{10}$!).

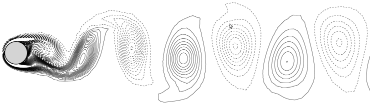
"In principle, concepts like coherent structures are best left implicit."

Hussain, "Coherent structures in a turbulent boundary layer", (1986) *Phys. Fluids.*

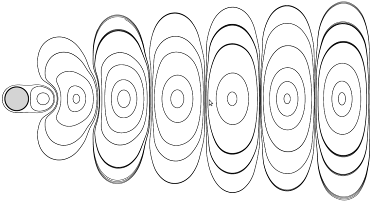
Several relevant frameworks exist to identify coherent structures.

Modal framework

The aim is to give a relevant representation of a dataset, *e.g.* the energy (POD) or the frequencies (Fourier).



Cylinder wake

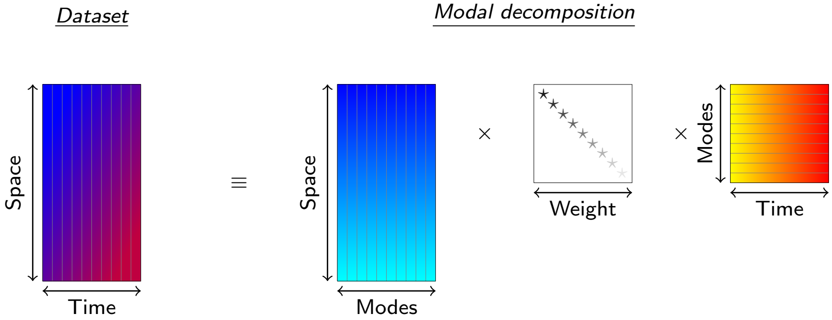


POD mode 1

Bergmann & Cordier, (2008) *J. Comput. Phys.*

Modal framework

The aim is to give a relevant representation of a dataset, *e.g.* the energy (POD) or the frequencies (Fourier).



It may lead to model reduction, through Galerkin-projection or truncature.

Lagrangian framework

Eruptions of Eyjafjallajökull. Estimated ash cloud on 15 April 2010.

In fluid mechanics :

$$\dot{\boldsymbol{X}} = \boldsymbol{u}(\boldsymbol{X}, t)$$

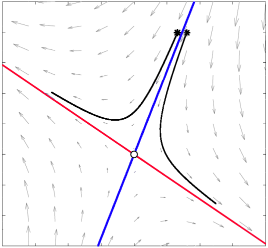
with $\boldsymbol{X} \in \mathcal{R}^3$

$\boldsymbol{u}$ comes from DNS or PIV measurement.

$\nabla \cdot \boldsymbol{u} = 0$ implies the system is conservative (within dynamical system frame).

If the system is autonomous or periodic, then the dynamic is driven by invariant manifolds.

Lagrangian framework



In autonomous system, stable and unstable manifolds :

- are attached to a fixed point
- are invariant
- are material frontiers
- are edges of invariant sets
- drive transport and mixing
- are hyper planes of locally maximum stretching

Big data

Understand fluid mechanics

- Numerous fields/points of view
  - Velocity
  - Pressure
  - Temperature
  - Concentration ...
- Large 3DnC simulations
- Hi-Res experimental snapshots

Leads to huge dataset

- number of points : $c \times n^d \times N$

with typically

	c	d	N	n
DNS	5	3	1000	124
Exp	2	2	10000	1000

How to **efficiently** identify coherent structures and/or the **most relevant** components from such a dataset ?

$\rightsquigarrow$ Observability and uncertainties have to be quantified.

My work so far

- Identification of most observable points or most relevant points w.r.t. the dynamics
- Computations of coherent structures in case of large/incomplete/non-uniform dataset
- Hash functions, stochastic model and control from sparse observable
- Mixing : ROM and change of paradigm for efficient computations
- Data assimilation, UQ and diagnostic of accuracy
- Exploration of a cavity flow properties.

Aims of this presentation

- Identifying relevant points w.r.t. the dynamics
  - most observable points
  - most representative points
- Spectral decomposition with ill-conditioned dataset
- Brief overview of :
  - Stochastic model and control of a system possibly unknown
  - Characterizing areas of mixing in turbulent flows with ROM.
  - Diagnostic in 4DVAR

Outline

- Intro
- Spectral decomposition of a dataset
  - Dynamic Mode Decomposition
  - Non-Uniform DMD
- Data-driven observability
- Mixing
- Data-driven statistical methods for ROM and control
- Diagnostic in 4DVAR
- Outro

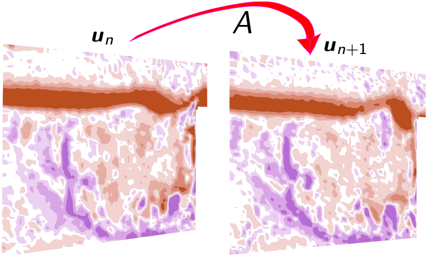
– Dynamic Modes Decomposition (DMD) –

"We represent a fluctuating signal by the mean (time-averaged) contribution, the periodic wave and the turbulent motion. ."

Reynolds & Hussain, "The mechanics of an organized wave in turbulent shear flow", (1972) *J. Fluid Mech.*

What are dynamic modes ?

Schmid[1] ; Rowley[2] ;
$\rightarrow$ Assume there exists an operator of evolution, A, such as the $\boldsymbol{u}_k$ are realisations of a *nonlinear* process.



$\rightarrow$ Find a similar matrix to A. *Dynamic modes are defined as eigenvectors of A*, computed thanks to the similar matrix.

1. Schmid *et al* (2008) *66th APS meeting* ; Schmid, (2010) *J. Fluid Mech.*
2. Rowley *et al*, (2009) *J. Fluid Mech.*

Defining the Evolution Operator A [PoF2014]

If ϕ is the flow of the fluid dynamical system :

$$\boldsymbol{X}_{n+1} = \phi_{\Delta t} \boldsymbol{X}_n,$$

and Π is the projector onto the experimental space (*i.e.* $\boldsymbol{u}_n = \Pi \boldsymbol{X}_n$), A is defined by :

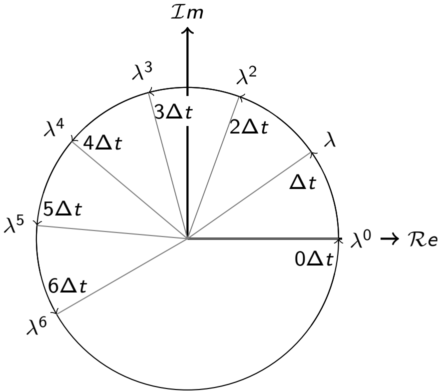
$$A \circ \Pi = \Pi \circ \phi_{\Delta t}.$$

Then,

$$\begin{aligned} A\boldsymbol{u}_n &= A \circ \Pi \boldsymbol{X}_n \\ &= \Pi \circ \phi_{\Delta t} \boldsymbol{X}_n \\ &= \Pi \boldsymbol{X}_{n+1} \\ &= \boldsymbol{u}_{n+1} \end{aligned}$$

Spectral properties of DMD

$$\boldsymbol{u}_n = \sum_j a_i^1 \lambda_j^n \boldsymbol{\Phi}_i \equiv \sum_j \lambda_j^n \boldsymbol{\Phi}_i ,$$



which means :

$$\lambda = \rho \exp\left(\sqrt{-1}\omega\Delta t\right) .$$

DMD and uniform sampling

The DMD algorithm needs an uniform sampling.

Data problems

- Corrupted dataset
- Incomplete dataset
- Convergence of data pre/post-treatment

uniform sampling is not always possible

Experimental issues : example taken from Fluid Mechanics

Observable :
2D2C field (PIV) $\rightarrow 1000 \times 1000px$
Frequencies of the flow :

1. one low ($\approx 0.1Hz$) $\Rightarrow$ $10s$ of sampling at least
2. one high ($\approx 200Hz$)$\Rightarrow$ sampling rate at $400Hz$

Depth of images : 12-bit
Broad-band needed :
$bb = 400 \times 1000^2 \times 12 > 4Gb.s^{-1}$
for at least $10s$

Unreachable for standard material

Non-Uniform DMD [ICTAM2012,PoF2015]

With the expression

$$\boldsymbol{u}_n = \sum_j a_i^1 \lambda_j^n \boldsymbol{\Phi}_i \equiv \sum_j \lambda_j^n \boldsymbol{\Phi}_i,$$

we can write more generally :

$$\begin{array}{lllll} \boldsymbol{u}_{t_n} & = & \sum_j \lambda_j^{t_n} \boldsymbol{\Phi}_j + \boldsymbol{e} & \approx & \lambda_1^{t_n} \boldsymbol{\Phi}_1 + \lambda_2^{t_n} \boldsymbol{\Phi}_2 + \dots \\ K & = & M\,V + R & \approx & M\,V. \end{array}$$

How to achieve this decomposition ?

$$K = M\,V + R \approx M\,V.$$

Pseudo-Vandermonde Matrix and Modes

V [3] is :

$$V = \begin{pmatrix} \lambda_1^{t_1} & \lambda_1^{t_2} & \dots & \lambda_1^{t_N} \\ \lambda_2^{t_1} & \lambda_2^{t_2} & \dots & \lambda_2^{t_N} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{N_{\mathrm{md}}}^{t_1} & \lambda_{N_{\mathrm{md}}}^{t_2} & \dots & \lambda_{N_{\mathrm{md}}}^{t_N} \end{pmatrix},$$

and M is the modes :

$$M = \left(\psi_1 \dots \psi_{N_{\mathrm{md}}}\right).$$

3. times t_i are taken arbitrary, not necessary ordered.

How to achieve this decomposition ?

Obtaining of the Spatial Modes

Matrix M is easily computed :

$$M \approx K\, V^{+},$$

where V^{+} is Moore-Penrose pseudo-inverse of V.

Obtaining the frequencies [PoF2015]

Compressed computing

A low number of modes is supposedly dominant $\rightsquigarrow$ Temporal spectrum of the system is sparse.

- Compressed sensing approach [4] .
  - $\rightsquigarrow$ N_{md} modes are chosen.
  - $\rightsquigarrow$ only $\widetilde{N} \geq 2N_{\text{md}}$ are necessary.
- Clustering [5] components with similar spectral features, based on the sparse spectrum.
  - $\rightsquigarrow$ Select $\tilde{n}_p \ll n_p$ ones.

$\rightsquigarrow$ K is replaced by $\widetilde{K} \in \mathbb{R}^{\tilde{n}_p \times \widetilde{N}}$

4. D.L. Donoho *et al.*, (2006) *IEEE T. Inform. Theory*
5. n_p is the size of the observable, and K is $n_p \times N$

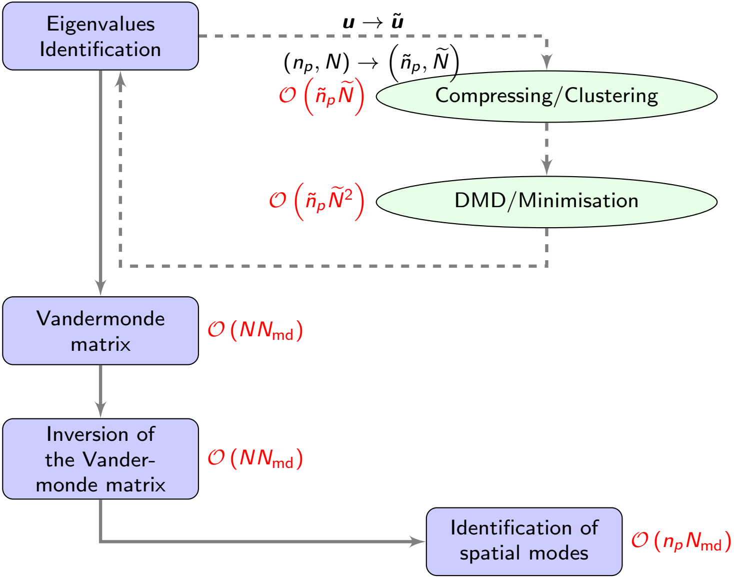
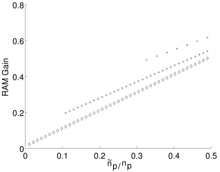
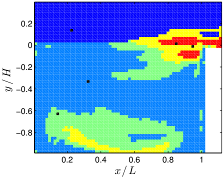

Eigenvalues Identification
$\boldsymbol{u} \rightarrow \tilde{\boldsymbol{u}}$
$(n_p, N) \rightarrow \left(\tilde{n}_p, \widetilde{N}\right)$
$\mathcal{O}\left(\tilde{n}_p \widetilde{N}\right)$
Compressing/Clustering
$\mathcal{O}\left(\tilde{n}_p \widetilde{N}^2\right)$
DMD/Minimisation
Vandermonde matrix
$\mathcal{O}\left(N N_{\mathrm{md}}\right)$
Inversion of the Vandermonde matrix
$\mathcal{O}\left(N N_{\mathrm{md}}\right)$
Identification of spatial modes
$\mathcal{O}\left(n_p N_{\mathrm{md}}\right)$

Illustration [ICTAM2012,PoF2015]

Efficiency of compressed approach





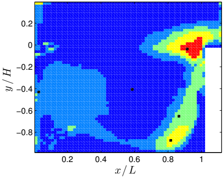
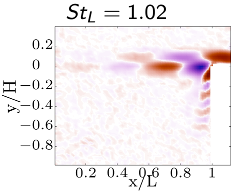
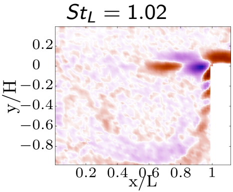


Illustration [ICTAM2012,PoF2015]

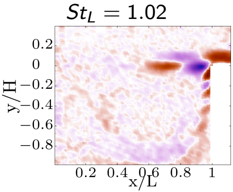
Results on the cavity flow



DMD mode
500 snapshots
7000 spatial points



NU-DMD mode
10 randomly taken snapshots
7000 spatial points



NU-DMD mode
10 randomly taken snapshots
10 spatial points

– Observability –

If we consider time series $\tilde{\boldsymbol{u}}$ extracted from $\boldsymbol{u}$[6], we can write the automatic system :

$$\begin{cases} \boldsymbol{u}_{n+1} & = & A\boldsymbol{u}_n \\ \tilde{\boldsymbol{u}}_n & = & C\boldsymbol{u}_n \end{cases},$$

Then the system is observable ($\boldsymbol{u}_0$ can be reconstructed from the n_p first $\tilde{\boldsymbol{u}}$) if the Kalman matrix[7] has a full rank :

$$\mathcal{K} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n_p-1} \end{pmatrix}.$$

Practically, the conditioning of A^i is blowing up, so it is undoable to estimate the observability qualities of time series with the Kalman Matrix.

6. where $\boldsymbol{u} \in \mathcal{R}^{np}$
7. Kalman, "Contributions to the theory of optimal control", Bol. Soc. Mat. Mexicana (1960)

" When you think about a variable, the evolution of it must be influenced by whatever others variables it's interacting with. Their values must somehow be contained in the history of that thing. Somehow their mark must be there. "

James Farmer, (1986) *Interview with James Gleick*

J.D. Farmer *et al.*, "Geometry from a time series", (1980), *PRL*

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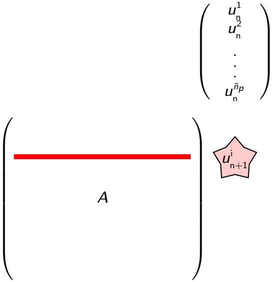
Propagation of the field

$\boldsymbol{u}_{n+1} = A\boldsymbol{u}_n$:



The ith line allows to count components **influential** in the dynamics of the observable $\{u^i\}$.
The ith column allows to count components whose dynamics **is influenced** by the observable $\{u^i\}$.

Observability criterion [RNL2013]

Then, by counting the number of significant components of A :

- n_l on the ith line $\equiv$ *quality*
- n_c on the *ith* column $\equiv$ *scarcity*

we define the DMD-observability for the ith component as :

$$\sigma_\alpha(i) = \frac{1}{n_p}\left(\alpha n_l + (1-\alpha)(n_p - n_c)\right)$$

We can approximate the operator A [8] , thanks to the DMD algorithm :

$$A \approx \widehat{A} = K_1^N S K_1^{N^{-1}}.$$

8. $\widehat{A}$ is the approximation of A in the span of the snapshots.

Toy examples

If we take a synthetic matrix $A \in \mathcal{M}_n$, and a random test vector $v \in \mathcal{M}_{n,1}$, we can construct a synthetic dataset :

$$K_1^N = \left\{ A \times v, A^2 \times v, \ldots, A^N \times v \right\}$$

Toy examples

If A is :

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

then :

Component :	1	2	3	4	5
Rank of $\mathcal{K}$	3	1	4	3	3
$\sigma_{0.5}$	0.48	0.44	0.60	0.50	0.47

Toy examples

If A is :

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & \color{red}{1} & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

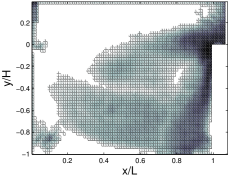
then :

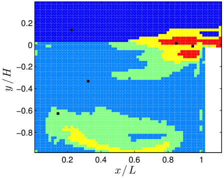
Component :	1	2	3	4	5
Rank of $\mathcal{K}$	3	1	5	3	3
$\sigma_{0.5}$	0.56	0.30	0.64	0.49	0.51

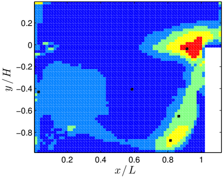
Illustration on well-known systems [RNL2013]

System	Observability order	Componants	Criterion
Röessler	$y \rhd x \rhd z$	y	0.79 ± 0.02
		x	0.77 ± 0.01
		z	0.09 ± 0.01
Lorenz	$z \rhd x \rhd y$	z	0.96 ± 0.01
		x	0.09 ± 0.00
		y	0.22 ± 0.01
Lorenz'84	$x \rhd y \approx z$	x	0.85 ± 0.01
		y	0.76 ± 0.01
		z	0.76 ± 0.01

Illustration on a cavity flow [RNL2013]







Good agreement with experimental placement of sensors and with Basley [9] .

9. J. Basley, 2012, *PhD thesis*,2014,*J.Fluid Mech.*

– Sparse modelling –

"Why go to so much effort to acquire all the data when most of what we get will be thrown away ?"

Donoho, "Compressed Sensing", (2006) *T. Inform. Theory*

Representation from a sparse observable[TCFD2016]

A hash function $\mathfrak{h} : \mathcal{R}^{n_e} \rightarrow \mathcal{N}$ associates an entry $\boldsymbol{y}$ with a *key* k.

$$h^{\boldsymbol{v},w}\left(\boldsymbol{y}\right) := h_0 + \left\lfloor \frac{\boldsymbol{v} \cdot \boldsymbol{y}}{w} \right\rfloor .$$

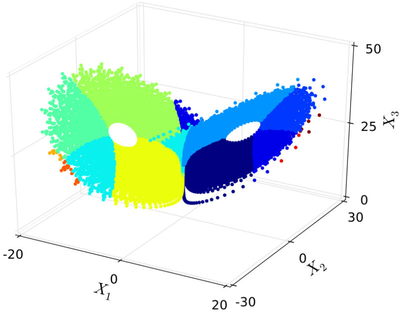
It can applied to any (sparse) observable y, embedded as

$$\boldsymbol{y} \equiv \left(y(t - \Delta t) \ldots y(t - (n_e - 1)\Delta t)\right),$$

- ✓ every kind of observable (even 1-D)
- ✓ Very computational friendly

Representation from a sparse observable[TCFD2016]

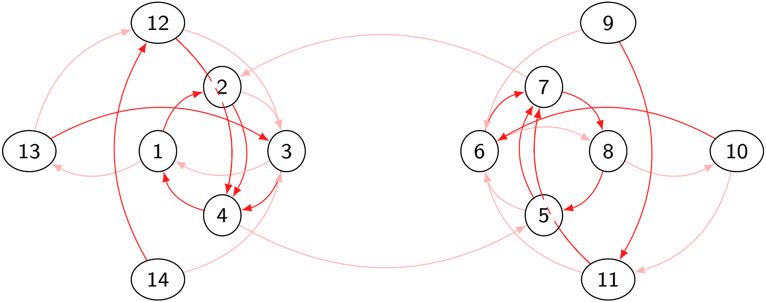
A hash function $\mathfrak{h} : \mathcal{R}^{n_e} \to \mathcal{N}$ associates an entry $\boldsymbol{y}$ with a *key* k.



Keys (*i.e.*, objects in the image space of $\mathfrak{h}$) generate a Voronoï paving of the observable space.

Modelling the dynamics

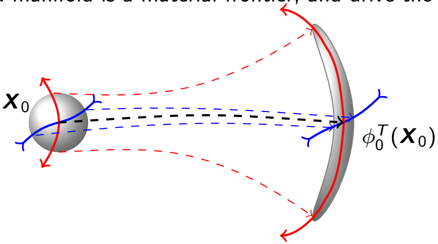
Based on transition probabilities from clusters to clusters, a stochastic model can be derived.



– Identification of Lagrangian Structures –

Material frontier

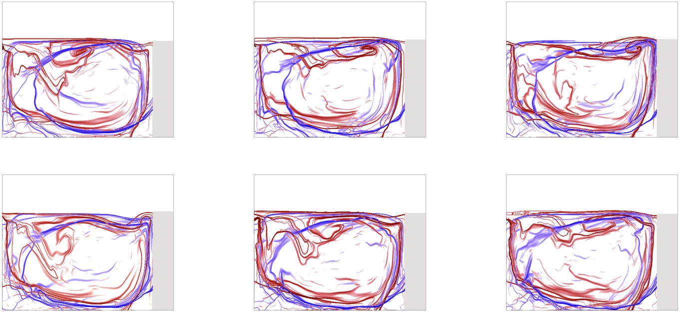
Dynamical flow : $\vec{\boldsymbol{X}}(t) = \phi^t\left(\vec{\boldsymbol{X}}(0)\right)$
Invariant manifolds are invariant w.r.t. the flow.
Consequently : such a manifold is a material frontier, and drive the mixing.



temps : $t = 0$ temps : $t = T$

Forward and backward LCS

Qualitative results



"These intersections will be seen forming a kind of lattice. (...) One will be startled by the complexity of this figure."

Henri Poincaré, "Sur les méthodes nouvelles de la mécanique céleste" (1899) *Dover*

Simplification of the flow

DMD and Restricted Reduced Order Model

We build a synthetic flow, based on a Dynamical Modes Decomposition analysis of the dataset :

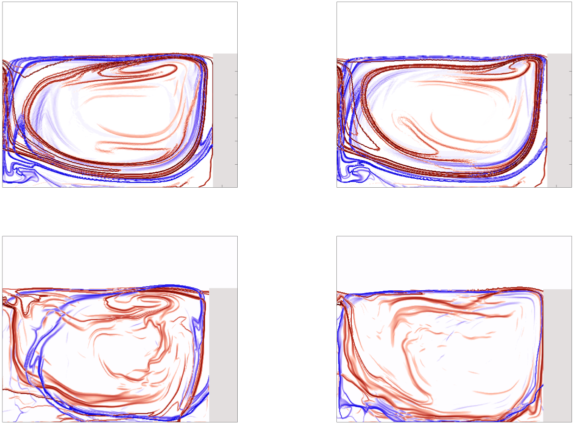
$$\boldsymbol{u_{ROM}}\left(\boldsymbol{r},t\right)=\bar{\boldsymbol{u}}\left(\boldsymbol{r}\right)+\mathcal{R}e\left(\mathrm{e}^{i\omega t}\boldsymbol{\Phi}_{\omega}\left(\boldsymbol{r}\right)\right)$$

where $\boldsymbol{\Phi}_{\omega}$ is the dominant shear layer DMD mode.
The periodicity of the model is important : Now one can see LCS structures as invariant manifolds in a Poincarré' section.

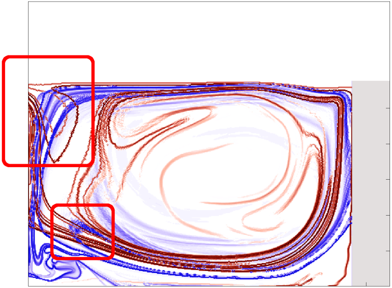
Verification of relevance

Comparison with real flows

ROM flow structures (top) and real flow structures (bottom), with similar horizon.



Horse-Shoes



– Diagnostic for Data Assimilation –

"It is particularly desirable to take the best possible advantage of available observations (...) to define initial conditions and produce a description of the state."

O. Talagrand, "Assimilation of Observations", (1997) *J. Meteorol. Soc. Jpn*

What is data assimilation ?

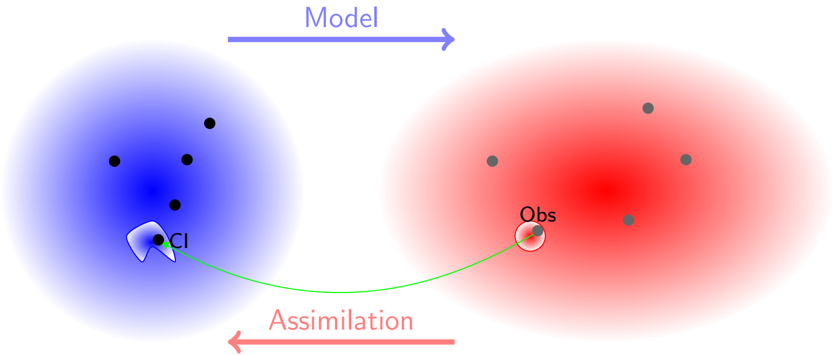
Consider the functional $\mathcal{J}$ [10] :

$$\mathcal{J}(\boldsymbol{u}_0, q) = \frac{1}{2}\sum_{i=0}^{N}(H(\boldsymbol{u}_i) - y_i)^t(H(\boldsymbol{u}_i) - y_i)$$

where H is the observational operator.
The minimizer of $\mathcal{J}$ are the "best" initial conditions $\boldsymbol{u}_0$ and parameters q
$\rightsquigarrow$ The model will produce fields as compatible as possible with the observations.

10. Covariance and background terms have been omitted.

What is data assimilation ?



Noises in observations

Practically, the observations are not perfect :
$\hat{y}_i$ are available, where $\hat{y}_i = H(\boldsymbol{u}_i) + \eta_i = y_i + \eta_i$.
It means that, the constructed cost function $\widehat{\mathcal{J}}$ is actually :

$$\widehat{\mathcal{J}}(\boldsymbol{u}_0, q) = \frac{1}{2}\sum_{i=0}^{N}(H(\boldsymbol{u}_i) - \hat{y}_i)^t(H(\boldsymbol{u}_i) - \hat{y}_i).$$

Diagnostic

The discrepancy between the approximate and the perfect cost function is :

$$
\begin{aligned}
\widehat{J}(\boldsymbol{u}_0, q) &= \tfrac{1}{2}\sum_{i=0}^{N}(H(\boldsymbol{u}_i) - \hat{y}_i)^t(H(\boldsymbol{u}_i) - \hat{y}_i) \\
&= \tfrac{1}{2}\sum_{i=0}^{N}(H(\boldsymbol{u}_i) - y_i - \eta_i)^t(H(\boldsymbol{u}_i) - y_i - \eta_i) \\
&= \tfrac{1}{2}\sum_{i=0}^{N}(H(\boldsymbol{u}_i) - y_i)^t(H(\boldsymbol{u}_i) - y_i) = \mathcal{J}(\boldsymbol{u}_0, q) \\
&\qquad -\tfrac{1}{2}\sum_{i=0}^{N}\eta_i^t(H(\boldsymbol{u}_i) - y_i - \eta_i) + (H(\boldsymbol{u}_i) - y_i - \eta_i)^t\eta_i \\
&\qquad +\tfrac{1}{2}\sum_{i=0}^{N}\eta_i^t\eta_i.
\end{aligned}
$$

An approximation of the noise's realization in the observations is, using the Johnson-Lindenstrauss lemma :

$$\tilde{\eta}_i = \hat{y}_i - H(\hat{\boldsymbol{u}}_i) \approx \eta_i$$

If the statistics of $\tilde{\eta}_i$ are different from the expected statistics from η_i, then the minimizer is, most probably, inaccurate, and should be used with care.

– Conclusions –

Conclusions

- Model and control from sparse observations[1].
- Methods for extracting spectral informations of the dynamics, fitted for ill-conditionned dataset[2–4].
- Identification of representative/observable points[2,5]
- FTLE field computations[6].
- Diagnostic for 4DVAR [7].

Some publications

1. *A statistical learning strategy for closed-loop control of fluid flows*, TCFD, 2016 (in revision)
2. *A Dynamic Mode Decomposition approach for large and arbitrarily sampled systems*, PoF 27, 2015
3. *Investigating mode competition and 3D features from 2D velocity fields in an open cavity flow by modal decompositions*, PoF 26, 2014
4. *POD-Spectral Decomposition for Fluid Flow Analysis and Model Reduction*, TCFD 27, 2013
5. *DMD économique pour l'identification de structures dans des écoulements 3D*, RNL 2013
6. *GPU and SIMD Acceleration for Identification of Lagrangian Coherent Structures. Application to an Open Cavity Flow*, BIFD 2013
7. *On the effect of the uncertainties on the distribution of initial conditions*, Tallus (in prep.)

More infos available on my website :

`www.gueniat.fr/publications.html`

Thank you for your attentiveness.
If you have any questions, I will be pleased to answer those.