Non-Uniform Dynamic Mode Decomposition

Snapshot-based Flow Analysis with Arbitrary Sampling

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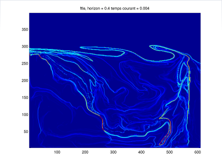


Hydrodynamic systems

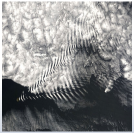
Complex systems...

Both open and confined flows are complex, and has potentially an infinite number of DoF.
Even if Navier-Stokes equation is long-known, it is actually not fully understand. NS is still a **Millenium Problem**.

... But coherent structures !

Nevertheless, coherent structures (vortex, soliton, saturated instabilities) are present and it looks like it drives the dynamic !
⇝ A hint of a low dimensional central manifold ?
⇝ Possibility to define a relevant topology for the dynamic ?
It seems legit to look after a 'low' number of modes.

Von 'Heartman' street
Isla Socorro ($Re > 10^{10}$!).



Orographic Waves
Amsterdam Island.

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Actual dimension reduction

How to catch coherent spatial features ?

Suppose space and time separation and assume the existence of a basis, $\{\boldsymbol{\psi}_k(\boldsymbol{r})\}$ or $\{\alpha_k(t)\}$, to describe any flow realization

$$\boldsymbol{v}(\boldsymbol{r},t) = \sum_{k\geq 1} \alpha_k(t)\boldsymbol{\psi}_k(\boldsymbol{r}) \simeq \sum_{j=1}^{N_m} \alpha_k(t)\boldsymbol{\psi}_k(\boldsymbol{r})$$

Modal decomposition(s)

→ **Proper Orthogonal Decomposition** $(\alpha_k(t), \psi_k(\boldsymbol{r}))$
energy ranking empirical (orthogonal) structures computed from a correlation matrix of the snapshots, $U = \Psi\Sigma A^t$
→ but dynamically relevant modes could be non energetic

→ **Global modes** $(\omega, \psi_\omega(\boldsymbol{r}))$
destabilizing a base-flow in a linear stability analysis
→ But you need to know the operator of evolution $\dot{V} = UV$...

→ **Dynamical modes** $(\omega, \psi_\omega(\boldsymbol{r}))$ if global organisation/narrow-banded spectra characteristic of the non-linear regime $V_{n+1} = UV_n \equiv V_nC$
→ But you need time-resolved data...

Outline

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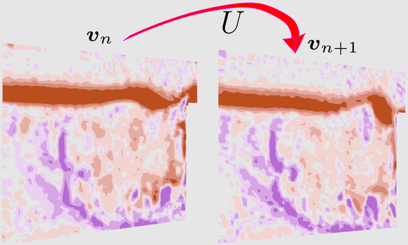
DMD

What are dynamic modes ?

Schmid *et al* (2008) *66*th *APS meeting* ; Rowley *et al*, (2009) *J. Fluid Mech.* ; Schmid, (2010) *J. Fluid Mech.*

→ Assume there exists an operator of evolution, U, such as the $\boldsymbol{v}_k$ are realisations of a *nonlinear* process.



→ Find a similar matrix to U. **Dynamic modes are defined as eigen-modes of U**, computed thanks to the similar matrix.

Companion matrix approach

With $\boldsymbol{v}_{N+1} = \boldsymbol{v}_1 c_1 + \ldots + \boldsymbol{v}_N c_N$, then :

$$AK_1^N = K_1^N C = K_2^{N+1} \qquad \Rightarrow \qquad C = \begin{pmatrix} 0 & 0 & \ldots & 0 & c_1 \\ 1 & 0 & \ldots & 0 & c_2 \\ 0 & 1 & \ldots & 0 & c_3 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \ldots & 1 & c_N \end{pmatrix}$$

Eigenvalues of C are eigenvalues of U. Let ν be an eigenvector associated with the eigenvalue λ :

$$\begin{aligned} UK_1^N \nu &= K_1^N C \nu \\ &= K_1^N \lambda \nu \\ &= \lambda K_1^N \nu \\ U \left(K_1^N \nu \right) &= \lambda \left(K_1^N \nu \right) \end{aligned}$$

Eigenvectors ψ of U are derived from eigenvectors ν of U : $\boldsymbol{\psi} \equiv K_1^N \boldsymbol{\nu}$

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Non-Uniform Dynamic Mode Decomposition

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Context 1/3

Data problems

- Corrupted dataset
- Incomplete dataset
- Convergence of data pre/post-treatment

uniform sampling is not always possible

Experimental problems : example taken in Fluid Mechanics

Observable :

2D2C field (PIV) $\rightarrow 1000 \times 1000px$

Frequencies of the flow :

1. one low ($\approx 0.1Hz$) $\Rightarrow$ $10s$ of sampling
2. one high ($\approx 200Hz$)$\Rightarrow$ sampling rate at $400Hz$

Depth of images : 12-bit
Broad-band needed :
$bb = 400 \times 1000^2 \times 12 > 4Gb.s^{-1}$
for at least $10s$ unreachable for standard material.

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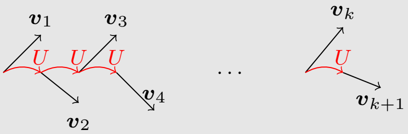
Context 2/3

DMD needs a uniform sampling.

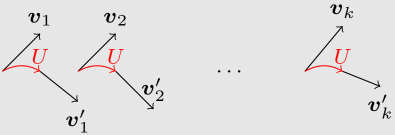
$$UK_1^N \equiv \{U\boldsymbol{v}_1, U\boldsymbol{v}_2, \ldots, U\boldsymbol{v}_N\} = \{\boldsymbol{v}_2, \boldsymbol{v}_3, \ldots, \boldsymbol{v}_{N+1}\} \equiv K_2^{N+1}$$



Context 3/3

Therefore, how to extract modal and temporal information from a non-uniform sampled dataset ?

$U K_1 \equiv \{U \boldsymbol{v}_1, U \boldsymbol{v}_2, \ldots, U \boldsymbol{v}_N\} = \{\boldsymbol{v}'_1, \boldsymbol{v}'_2, \ldots, \boldsymbol{v}'_N\} \equiv K_2$



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Decomposition

If we have the dataset K_1 :

$$K_1 = \left(\boldsymbol{v}_{t_1} \; \boldsymbol{v}_{t_2} \; \ldots \; \boldsymbol{v}_{t_N} \right).$$

times t_i are taken arbitrary, not necessary ordered.
We seek for a Fourier-like representation :

$$\begin{aligned} \boldsymbol{v}_{t_i} &= \sum_{j=1}^{N_m} \lambda_j^{t_i} \boldsymbol{\psi_j}(\boldsymbol{r}) + \boldsymbol{e} &\approx \quad & \lambda_1^{t_i} \boldsymbol{\psi}_1 + \lambda_2^{t_i} \boldsymbol{\psi}_2 + \ldots + \lambda_{N_m}^{t_i} \boldsymbol{\psi}_{N_m} \\ K_1 &= M\, V + R &\approx \quad & M\, V \end{aligned} .$$

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How to achieve this decomposition ?

$$K = M\,V + R \approx M\,V.$$

Pseudo-Vandermonde Matrix and Modes

V is :

$$V = \begin{pmatrix} \lambda_1^{t_1} & \lambda_1^{t_2} & \dots & \lambda_1^{t_N} \\ \lambda_2^{t_1} & \lambda_2^{t_2} & \dots & \lambda_2^{t_N} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{N_m}^{t_1} & \lambda_{N_m}^{t_2} & \dots & \lambda_{N_m}^{t_N} \end{pmatrix},$$

and M is the modes :

$$M = \left(\boldsymbol{\psi}_1 \ \dots \ \boldsymbol{\psi}_{N_m}\right).$$

reminder : times t_i are taken arbitrary, not necessary ordered.

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How to achieve this decomposition ?

Obtaining of the Spatial Modes

Matrix M is easily computed :

$$M \approx K\,V^{+},$$

where V^{+} is Moore-Penrose pseudo-inverse of V.

Obtaining the frequencies

M can be switched in equation[1] $K = M\,V + R$. Then :
$K \approx K\,V^{+}\,V + R$.

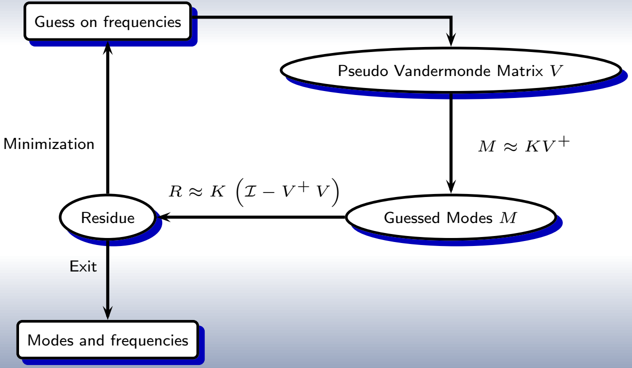
V can be computed by minimizing the residue matrix R :

$$R \approx K\,\left(\mathcal{I} - V^{+}\,V\right).$$

The modes follows immediately through $M \approx K\,V^{+}$.

[1] K. Chen, J.H. Tu & C.W. Rowley, Variants of dynamic mode decomposition : connections between Koopman and Fourier analyses, *Journal of Nonlinear Science*, submitted, 2011.

NU-DMD Algorithm



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Results

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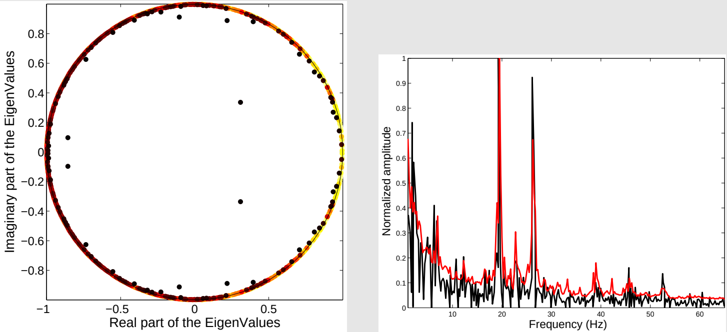
Illustration on a cavity flow

DMD : Eigenvalues of the similar matrix, associated spectrum



(a) (b)

FIGURE: (a) Eigenvalues (complex), (b) DMD Spectrum (black), build on the $\mathcal{L}_2$ norm of the dynamic modes.

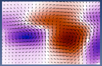
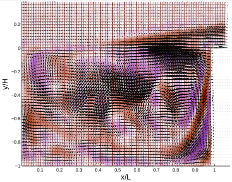
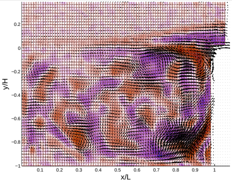


Illustration on a cavity flow

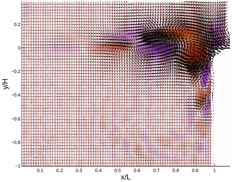
Dominant DMD modes



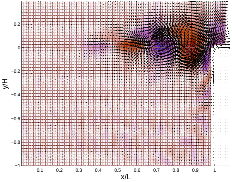
$f = 1.84$ Hz



$f = 6.30$ Hz



$f = 19.29$ Hz



$f = 26.11$ Hz

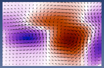


Illustration on a cavity flow

Comparison between DMD and NU-DMD

DMD needs N first snapshots of the dataset.
NU-DMD uses N snapshots taken at random in the same dataset.

N	12	28	38	65	90	200	500	1000
DMD (Hz)	20.8	26.8	26.3	26.9	27.8	27.5	27.0	27.0
NU-DMD (Hz)	27.0	27.0	27.0	27.0	27.0	27.0	27.0	27.0

TABLE: Frequencies of the main mode, computed by algorithms

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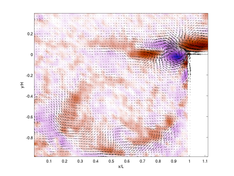
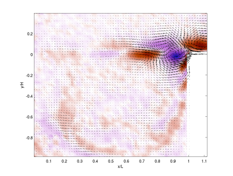
Illustration on a cavity flow

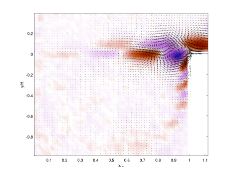
NU-DMD Mode
12 snapshots



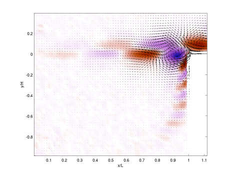
NU-DMD Mode
65 snapshots



NU-DMD Mode
500 snapshots



DMD Mode
500 snapshots



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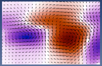
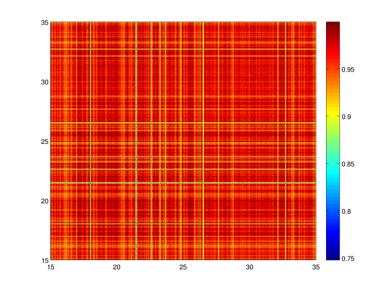


Illustration on a cavity flow

Drawback

Response surface (for 12 snapshots) is pretty hilly.



In non-smooth dataset, classical quick convergent methods for optimization will not work fully. A partial remedy is to consider an ensemble of initial guesses for the optimization step.

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Conclusions and openings

NU-DMD Algorithm can deal with deprecated, corrupted or non uniformly sampled dataset.
It can be used in large dataset with arbitrary sampling, the algorithm perform well to identify frequencies and spatial modes where Discrete Fourier Transform or DMD algorithm fail.

Nevertheless, the price to paid is the computation costs. In case of small **and** non-smooth dataset, only heavy methods – with large ensemble of initial guesses– **for now**, can achieve the minimization needed.

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This is the end

Thank you for your patience and attentiveness !

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